

UNITED STATES  
DEPARTMENT OF  
COMMERCE  
PUBLICATION



# ESSA Technical Memorandum WBTM TDL 33

**U.S. DEPARTMENT OF COMMERCE**  
ENVIRONMENTAL SCIENCE SERVICES ADMINISTRATION  
**Weather Bureau**

## Calculation of Precipitable Water

L. P. HARRISON

**Systems  
Development Office**

SILVER SPRING  
MARYLAND

June 1970

ESSA TECHNICAL MEMORANDA

Weather Bureau, Techniques Development Laboratory Series

The primary purpose of the Techniques Development Laboratory of the Office of Systems Development is to translate increases in basic knowledge in meteorology and allied disciplines into improved operating techniques and procedures. To achieve this goal, TDL conducts and sponsors applied research and development aimed at improvement of diagnostic and prognostic methods for producing weather information. The laboratory carries out studies both for the general improvement of prediction methodology used in the National Meteorological Service System and for more effective utilization of weather forecasts by the ultimate user.

ESSA Technical Memoranda in the Weather Bureau Techniques Development Laboratory series facilitate rapid distribution of material which may be preliminary in nature and which may be published formally elsewhere at a later date. The first five papers in the TDL series are part of the former Weather Bureau Technical Notes series.

Papers listed below are available from the Clearinghouse for Federal Scientific and Technical Information, U.S. Department of Commerce, Sills Bldg., 5285 Port Royal Road, Springfield, Va. 22151. Price: \$3.00 hard copy; \$0.65 microfiche. Order by accession number shown in parentheses at end of each entry.

- TN 10 TDL 1 Objective Prediction of Daily Surface Temperature. William H. Klein, Curtis W. Crockett, and Carlos R. Dunn, October 1965. (PB-168 590)
- TN 11 TDL 2 Hurricane Cindy Galveston Bay Tides. N. A. Pore, A. T. Angelo, and J. G. Taylor, September 1965. (PB-168 608)
- TN 29 TDL 3 Atmospheric Effects on Re-Entry Vehicle Dispersions. Karl R. Johannessen, December 1965. (PB-169 381)
- TN 45 TDL 4 A Synoptic Climatology of Winter Precipitation from 700-mb. Lows for the Intermountain Areas of the West. D. L. Jorgensen, W. H. Klein, and A. F. Korte, May 1966. (PB-170 635)
- TN 47 TDL 5 Hemispheric Specification of Sea Level Pressure from Numerical 700-mb. Height Forecasts. William H. Klein and Billy M. Lewis, June 1966. (PB-173 091)
- WBTM TDL 6 A Fortran Program for the Calculation of Hourly Values of Astronomical Tide and Time and Height of High and Low Water. N. A. Pore and R. A. Cummings, January 1967. (PB-174 660)
- WBTM TDL 7 Numerical Experiments Leading to the Design of Optimum Global Meteorological Networks. M. A. Alaka and F. Lewis, February 1967. (PB-174 497)
- WBTM TDL 8 An Experiment in the Use of the Balance Equation in the Tropics. M. A. Alaka, D. T. Rubsam, and G. E. Fisher, March 1967. (PB-174 501)
- WBTM TDL 9 A Survey of Studies of Aerological Network Requirements. M. A. Alaka, May 1967. (PB-174 984)
- WBTM TDL 10 Objective Determination of Sea Level Pressure from Upper Level Heights. William Klein, Frank Lewis, and John Stackpole, May 1967. (PB-179 949)
- WBTM TDL 11 Short Range, Subsynchronous Surface Weather Prediction. H. R. Glahn and D. A. Lowry, July 1967. (PB-175 772)
- WBTM TDL 12 Charts Giving Station Precipitation in the Plateau States from 700-Mb. Lows During Winter. D. L. Jorgensen, A. F. Korte, and J. A. Bunce, Jr., October 1967. (PB-176 742)
- WBTM TDL 13 Interim Report on Sea and Swell Forecasting. N. A. Pore and W. S. Richardson, December 1967. (PB-177 038)
- WBTM TDL 14 Meteorological Analysis of 1964-65 ICAO Turbulence Data. DeVer Colson, September 1968. (PB-180 268)
- WBTM TDL 15 Prediction of Temperature and Dew Point by Three-Dimensional Trajectories. Ronald M. Reap, September 1968. (PB-180 727)

(Continued on inside back cover)

U. S. DEPARTMENT OF COMMERCE  
Environmental Science Services Administration  
Weather Bureau

ESSA Technical Memorandum WBIM TDL 33

CALCULATION OF PRECIPITABLE WATER

L. P. Harrison



Systems Development Office  
Techniques Development Laboratory

SILVER SPRING, MD.  
June 1970

UDC 551.501.771:551.571.7

551.5            Meteorology  
  .501           Methods of observation  
                  and computation  
      .771        Methods of observation and  
                  computation of humidity  
  .571           Humidity and moisture  
      .7           Upper atmospheric humidity

## PREFACE

This paper is an extract from a larger body of writing (a monograph) prepared by the author on the subject of "Automatic Processing of Rawinsonde Observations," designed to indicate the mathematical and physical relationships, together with the relevant procedures, governing the evaluation of the rawinsonde observations. Underlying that monograph was the assumption that an observer determines the temperature, pressure, relative humidity, and time corresponding to the significant levels selected by him from the radiosonde data recorded for the sounding, and that generally an automatic means is provided for recording the simultaneously observed radar data, including the slant range, elevation angle, and azimuth angle at the end of each whole minute of the sounding or at prescribed fractions of a minute.

The present paper deals with the mathematical and physical relationships which may be used for calculating the "precipitable water" pertaining to a delimited air column.

## CONTENTS

	Page
Preface . . . . .	iii
Abstract . . . . .	1
1. Introduction . . . . .	2
2. Notation . . . . .	9
3. Formulas of Method I for Computing Precipitable Water . . . . .	15
4. Formulas of Method II for Computing Precipitable Water . . . . .	18
5. Numerical Integration Based on Two, Three, and Four Data Points of Nonlinear Functions . . . . .	21
6. Summary of Integration Procedures Depending upon Total Number of Consecutive Data Points Available . . . . .	37
7. Conclusions . . . . .	39
References . . . . .	40
Annex 1 - Theory Underlying Formulas for Computing Precipitable Water . . . . .	50
Annex 2 - Outline of Method of Determining the Acceleration of Gravity $g$ as a Function of Latitude and Geopotential . . . . .	58

## CALCULATION OF PRECIPITABLE WATER

L. P. Harrison

## ABSTRACT

Two methods of calculating precipitable water are given in terms of data obtainable from a radiosonde or rawinsonde observation. Method I is based on the premise that aqueous vapor pressure (or dew point) and temperature are reported for determined geopotential levels or geometric altitudes with reference to mean sea level; while Method II is predicated on the basis that aqueous vapor pressure (or dew point) is reported for respective specified barometric pressure levels whose geopotentials are computed by means of the data obtained in the sounding. Thus, under Method I the precipitable water is calculated by numerical integration such that the variable of integration is either geopotential or geometric altitude, and under Method II it is calculated by such a procedure that the variable of integration is pressure. In either case one may choose to assume arbitrarily that the acceleration of gravity  $g$  is constant, thus implicitly giving rise to a small systematic error; or one may treat the problem in a more accurate manner by taking  $g$  as a known function of latitude and geopotential (or geometric altitude). Procedures covering all of these alternatives are presented. In practically all actual situations dealt with here, the function which is subject to numerical integration is nonlinear with respect to the variable of integration which, itself, is generally given for unequal intervals. Explicit numerical integration formulas are presented to cope with these situations. In order to operate with a reasonable degree of simplicity under such circumstances, we have designed the formulas to apply effectively as curve-fitting devices to cases involving either three or four consecutive points of the sounding. These points may refer to significant levels or mandatory (constant-pressure) levels which are evaluated for the sounding and are generally interwoven. By adding the cumulative contributions of water vapor content from progressively higher "partial columns" based on the aggregate due to three or four such consecutive points of the sounding, one can compute the total precipitable water for deep atmospheric columns.

## 1. INTRODUCTION

To envisage the concept of "precipitable water," it is desirable to consider all of the water vapor in a vertical column of atmosphere having a small cross section, with the base of the column on the earth's surface at a prescribed geographic location and the top at a specified level either in the free air or at the top of the atmosphere.

By definition, the "precipitable water," denoted by  $W$  in general, represents the depth of liquid water which would be deposited on the horizontal base of the specified vertical column of atmosphere if all of the water vapor held within the column were condensed and precipitated out.\*

Where the vertical column of atmospheric air contains hydrometeors (such as cloud or fog particles, raindrops, snowflakes, sleet pellets, hailstones, etc.), the calculation of  $W$  is based on the assumption that at whatever level hydrometeors exist they are replaced by moist air having a concentration of water vapor equal to that within the spaces between them.

We must emphasize that the data obtained from radiosonde observations do not actually refer to a vertical column as specified by the definition of precipitable water. The geographical coordinates of the station from which the radiosonde is released, where  $\phi$  = latitude and  $\lambda$  = longitude, also are assumed to be the geographic coordinates of the points in free air at which the moisture and other physical properties of the atmosphere are observed. But during the course of the sounding the probing device has a sloping and generally curved trajectory. In effect, the vertical column must be imagined as replaced by a filament whose axis coincides with the trajectory of the radiosonde and whose horizontal cross section is circular. Thus, the precipitable water  $W$ , calculated from the available data, represents a sampling through the filament extending from the launching station to the point in the

---

\*To obviate any misunderstanding, it will be stated that "precipitable water" does not include any liquid or solid water originally suspended or falling in the vertical column of atmosphere under consideration. Thus,  $W$  excludes the liquid or solid water content of clouds, fog, hail, rain, sleet, snow, etc.

free air where the trajectory of the balloon intersects the prescribed level. (say 500 mb) at the top of the assumed vertical column.

With this clarification in mind, we shall hereafter refer only to the vertical column. One may imagine the horizontal cross-section area,  $A_c$ , of the vertical column as having a particular value. For example,  $A_c = 1 \text{ m}^2$  at the earth's surface.

Also, it is desirable that the symbol  $W$  have appended subscripts to indicate the levels of the lower and upper limits of the vertical column. In the notation adopted here (see list of symbols on page 10) the first subscript refers to the lower limit where  $s$  indicates "surface" and the second refers to the upper limit, usually in terms of the pressure at the top of the column. For example,  $W_{s,5}$  = precipitable water relating to a vertical column whose base is at the earth's surface and whose top is at the 500-mb constant pressure level;  $W_{s,0}$  = the precipitable water relating to a vertical column whose base is at the earth's surface and whose upper limit is at the top of the atmosphere where the barometric pressure is equal to zero.

There are many applications for the liquid-depth equivalent of the water vapor content of a vertical column of atmospheric air where both limits of the column are in free air. The liquid-water depth equivalent for such a column will be termed the "partial precipitable water" denoted by  $W$ , with subscripts to indicate the lower and upper limits, usually in terms of the barometric pressures at those limits. For example,  $W_{7,5}$  = the water-depth equivalent of the water vapor in the vertical column of atmospheric air whose base is at the 700-mb level and whose top is at the 500-mb level; similarly,  $W_{85,5}$  = the water-depth equivalent of the water vapor contained in the vertical column of atmospheric air whose base is at the 850-mb level and whose top is at the 500-mb level. Thus, "partial precipitable water" is related to "precipitable water," as illustrated by the following examples:

$$W_{7,5} = (W_{s,5} - W_{s,7})$$

$$W_{85,5} = (W_{s,5} - W_{s,85})$$

$$W_{s,5} = (W_{s,85} + W_{85,7} + W_{7,5})$$

Two methods of calculating the precipitable water will be treated in this report; we shall refer to them as "Method I" and "Method II." In Method I we consider that at any given geopotential  $H$  or corresponding geometric altitude  $Z$  within the vertical column of atmospheric air there exists an aqueous vapor pressure  $e$  and a temperature  $T$ , such that their ratio  $e/T$  governs the absolute humidity  $\rho_v$ . Hence, in Method I one can compute  $W$  by integrating the absolute humidity  $\rho_v$  with respect to geometric altitude  $Z$  or by integrating the ratio  $\rho_v/g$  with respect to geopotential  $H$ , where  $g$  = acceleration of gravity. In Method II consider that at any given pressure  $P$  within the vertical column of atmospheric air there exists an aqueous vapor pressure  $e$ , such that the ratio  $e/P$  governs the specific humidity  $q$ . Hence, in Method II one can compute  $W$  by integrating the ratio  $q/g$  with respect to pressure  $P$ . Mathematical derivations supporting the foregoing statements and developing integral formulas for  $W$  are presented in annex 1. The denominator of the integrand involves the acceleration of gravity  $g$  because  $g$  is entailed in the relationship between  $Z$  and  $H$ . This relationship and others theoretically relevant to the problem are presented in annex 2.

The integral formulas representing  $W$  based on Method I are summarized on pp. 15-17, and those based on Method II are summarized on pp. 18-20.

Some procedures for the numerical integration of the above-mentioned formulas are outlined on pp. 21-36. These procedures are designed for cases where the spacing of abscissas pertaining to given adjacent ordinates is unequal. Integration techniques for non-equal spacings are justified because the data for both significant levels and constant pressure levels will be used.

Routine forecasting at time of writing calls for determination of the value of  $W_{s,5}$ ; that is, the precipitable water in the vertical column extending from the surface to the 500-mb level. Within this range the mean value of the acceleration of gravity over the globe is around  $9.8 \text{ m/sec}^2 \pm 0.03 \text{ m/sec}^2$  (see annex 2). Thus if, in calculating  $W_{s,5}$ , one assumes that  $g$  in the denominator of the integrand has the constant value  $9.8 \text{ m/sec}^2$ , the relative error will not exceed about 0.3 percent, which is about one order of magnitude smaller than the relative error in the observed relative humidities. This

will justify the conventionally made assumption that  $g$  is constant.

The integral formulas for computing  $W$ , shown in secs. 4 and 5, present alternative forms with respect to the treatment of  $g$ ; that is, one formula indicates how  $g$  enters the problem which becomes fairly complex when  $g$  is treated as a variable, and another formula indicates how the problem is simplified when one assumes that  $g = \text{constant}$ , based on an average value of the acceleration of gravity over the range of altitude and latitude encountered in normal operations. The information contained in annex 2 shows how  $g$  can be evaluated as a function of latitude and geopotential, for application in calculating  $W$  while taking  $g$  as a variable under the integral sign. The contingency of taking  $g$  as a variable must be allowed for should it ever be necessary to calculate  $W$  pertaining to a vertical column from the surface to an altitude much higher than the 500-mb level. For such a contingency or for ascertaining the error from assuming a constant value of  $g$ , the material in annex 2 will be useful.

#### Brief Review of Literature

When the vertical coordinate is height above sea level, it is natural to compute the precipitable water by integrating the absolute humidity (concentration of water vapor) with respect to height. This procedure was followed by Harrison (1931) [31], who published seasonal mean values of the water-depth equivalent of the water-vapor content of the atmosphere over the United States east of the Rocky Mountains.

When, however, the vertical coordinate is pressure, it is natural to calculate the precipitable water by integrating the specific humidity with respect to pressure. A formula for computing precipitable water on this basis was derived by Solot [99]. Haltiner and Martin [30] have presented a brief, essentially similar derivation. The problem was also discussed by Goldschmidt [27].

Tables to facilitate the calculation of precipitable water, especially for the case of a saturated pseudo-adiabatic atmosphere, were prepared by the Hydrometeorological Section of the Weather Bureau [41]. Showalter [93] developed a template to permit the determination of precipitable water for layers

between adjacent standard constant-pressure surfaces when the dew-point temperatures observed during a sounding are plotted on a pseudo-adiabatic chart. A nomogram was devised by Peterson [78] to facilitate the computation of precipitable water for layers between the surface and the 850-, 700-, and 500-mb constant-pressure surfaces, under the assumption that the dew-point temperature varies linearly with pressure between those levels.

The total precipitable water in a vertical column extending to the top of the atmosphere may be determined by means of infrared radiation absorption measurements, using the sun or moon as the radiation source. Among the investigators who have utilized or discussed this method may be cited Fowle [25]; Kimball and Hand [43]; Kimball [42]; Foster and Foskett [23]; Schüepp [89]; Lejay [46]; Vassy [114] Goldschmidt [27]; Murcra, Murcra and Williams [62]; McMurry [57]; Alavi-Nejad [2,3]; Foster, Volz and Foskett [24]; Gates [26]; King and Parry [44]; and Hutcherson [38].

Correlations between precipitable water and water-vapor or other parameters observed at the surface, such as the dew-point temperature, have been studied by Lejay [46]; Montefinale and Papée [59]; Papée and Montefinale [73]; Montefinale, Zawidzki and Papée [60]; Reitan [87]; Bolsenga [16]; Smith [97]; and Rao [80].

The diurnal variation of precipitable water was investigated by Alavi-Nejad (1963) [3] and McMurry [57] using an infrared radiation absorption technique.

Neumann [64], Solomatina [98] and Tunnell [111] considered various aspects of the diurnal or semi-diurnal variations of water-vapor pressure at the surface. This problem, with special reference to the Alpine regions under high-pressure conditions, was studied by Hastenrath [33].

Presentations of seasonal or monthly climatological data pertaining to precipitable water or related quantities were given by Harrison (1931) [31]; Shands [92]; Tunnell [111]; Bannon and Steele [6]; Reitan (1960) [85,86]; Starr, Peixoto and Crisi (1965) [104]; Ananthakrishnan, Selvam and Chellappa [4]; Sellers [91]; and Tuller [110]. A special study by Byers [19] relates to the different vertical distributions of water vapor in arid and humid regions.

Additional data sources are listed in some of the following references.

With particular reference to high-level, upper-air profiles of water-vapor content the work of the following investigators may be cited: Miller [58]; Gutnick [28]; Mastenbrook and Dinger [54]; Murcray, Murcray, Williams and Leslie [63]; Murcray, Murcray and Williams [61,62]; and Hutcherson [38]. The special problem of the water-vapor content of horizontal, upper-air paths has been considered by Gutnick [28]; Sissenwine and Gutnick [95]; and Martin [53].

There have been a considerable number of papers and reports devoted to various applications of precipitable water, particularly in the fields of quantitative precipitation forecasting, hydrology, and general circulation. Many of these publications present such data as monthly or seasonal mean values of precipitable water, and derived quantities such as meridional and zonal water vapor transfer, flux divergence of water vapor, and calculated parameters indicating the difference between precipitation and evaporation at the surface. Because of the range in scope covered by the publications there exists a great deal of overlap. It is, therefore, difficult to classify all of the relevant papers and reports into narrow, mutually exclusive, categories. In what follows we shall endeavor to indicate the principal subjects covered by the publications to which reference is made:

#### Quantitative Precipitation Forecasting

Spar [100]; Smagorinsky and Collins [96]; Younkin, Larue and Sanders [118]; Shuman and Hovermale [94].

#### Synoptic Applications of Precipitable Water

Neuwirth [65].

#### Water Vapor Transfer and Flux Divergence

Benton [9,10]; Hutchings [39,40]; Bannon, Matthewman and Murray [5]; Palmén and Vuorela [72]; Starr, Peixoto and Crisi [104]; Bock, Frazier and Welsh [15]; Hastenrath [34,35,36]; Rasmusson [82].

Water Vapor Transfer

Benton and Estoque [12]; Starr and White [107] Flohn and Oeckel [22]; Starr and Peixoto [102,103]; Manabe [51]; Peixoto [74]; Van de Boogaard [113]; Vuorela and Tuominen [115]; Crisi [20]; Hastenrath [36]; Peixoto and Crisi [75]; Peixoto and Obasi [76]; Travelers Research Center [109]; Rasmusson [83].

Flux Divergence

Starr and Peixoto [101]; Nyberg [66,67]; Lufkin [50]; Barnes [7]; Ferguson and O'Neill [21]; Rasmusson [84].

Hydrology, Evapotranspiration, Water Budget and Difference Between Precipitation and Evaporation

Benton, Blackburn, and Snead [11]; Benton, Estoque, and Dominitz [13]; Benton and Estoque [12]; Benton [9,10]; Sutcliffe [108]; Bradbury [17,18]; Starr and Peixoto [101,102,103]; Manabe [51]; Starr, Peixoto and Livadas [105]; Nyberg [66,67] Lufkin [50]; Wilson [117]; Benton et al. [11]; McDonald [56]; Palmén and Holopainen [70]; Väisänen [112]; Huff [37]; Manabe, Smagorinsky and Strickler [52]; Peixoto and Obasi [76]; Sellers [91]; Palmén and Soderman [71]; Palmén [69]; Adem [1]; Rasmussen [81]; Rasmusson [82].

Energy Budget and General Circulation

Priestley [79]; White [116]; Starr and White [106]; London [48]; Peixoto and Saltzman [77] Obasi [68]; Starr and Peixoto [103] Hastenrath [35]; Lorenz [49]; Manabe, Smagorinsky, and Strickler [52].

Cloud Seeding Criterion

Battan [8].

Annotated Bibliography

Miller [58]; Kiss [45].

## 2. NOTATION

Presented here is a list of the various symbols employed, together with specifications, simple definitions, and units relating to them. All of the entities indicated are involved with the theoretical and practical problems of calculating precipitable water within delimited vertical columns of atmospheric air.

Under the assumption outlined in the introduction we ascribe the dependent variables (such as  $e$  = vapor pressure,  $T$  = temperature,  $g$  = acceleration of gravity, etc.) to the geographic coordinates  $(\phi, \lambda)$  of the station, and imply that the level to which those dependent variables relate is indicated by the vertical coordinate specified, in terms of the geometric altitude  $Z$ , geopotential  $H$ , or pressure  $P$ . Thus the vertical coordinate serves as the independent variable, while the latitude is an assumed constant parameter. Subscript s on symbols such as  $Z$ ,  $H$ ,  $P$  indicates that the data pertain to the surface (base of column), whereas subscript t indicates that the data relate to the top of the column. When using this notation we make clear that symbols  $Z_s$ ,  $H_s$ ,  $P_s$  represent the vertical coordinate referring to the lower limit of the column, and that symbols  $Z_t$ ,  $H_t$ ,  $P_t$  represent the vertical coordinate referring to the upper limit of the column.

The formulas for the calculation of precipitable water  $W$  requires numerical integration which generally involves either  $H$  or  $P$  as the independent variable, and with different integrands depending upon whether  $H$  or  $P$  is used. Nevertheless, the rules of numerical integration apply precisely in the same manner to both of these cases; hence we find it advantageous to generalize the notation employed in such integration. For this reason we use  $x$  to represent the independent variable ( $H$  or  $P$ ), and  $y$  to represent the integrand in the appropriate form depending upon whether  $H$  or  $P$  is the independent variable.

In terms of the generalized system of coordinates  $(x, y)$  employed in the numerical-integration formulas, the lower limits of the integrals are written  $x_s$  instead of  $H_s$  or  $P_s$ , while the upper limits of the integrals are written  $x_t$  instead of  $H_t$  or  $P_t$ .

The system of units employed includes the meter for length, the gram for mass, and the second for time. In the final forms of the integral expressions used for the calculation of  $W$ , the geopotential  $H$  is expressed in geopotential meters (gpm) and the pressure  $P$  in millibars. But in the derivation of the formula for  $W$  involving  $P$  as the independent variable (see annex 1),  $P$  is first expressed in the meter-gram-second system and later converted to millibars.

Symbols for the various entities are introduced when they first appear, and the following list is alphabetical for ready reference:

- $A$  = coefficient in Eq. (5.12), defined by Eq. (5.13).
- $A_c$  = cross section area of vertical column; see Eqs. (A1) and (A2).
- $b$  = exponent in Eq. (5.6), defined by Eq. (5.7) or (5.8).
- $B$  = coefficient in Eq. (5.12), defined by Eq. (5.14).
- $c$  = numerical constant =  $0.37802 = 1 - (M_v/M_a)$ ; see Eq. (4.3).
- $C$  = symbol designating the coefficient of one of Eqs. (3.1), (3.2), (4.1), or (4.2), whichever is appropriate for the application; see, for example, Eqs. (5.2), (5.5), (5.9), (5.13) - (5.15), and (5.19) - (5.22).
- $dm$  = infinitesimal mass of water vapor (see  $m$  = mass and  $dm$  following it).
- $D$  = coefficient in Eq. (5.12), defined by Eq. (5.15).
- $e$  = partial pressure due to water vapor in moist air, in millibars (mb).
- $E$  = coefficient in Eq. (5.18), defined by Eq. (5.19).
- $F$  = coefficient in Eq. (5.18), defined by Eq. (5.20).
- $F_i$  = functional indicator to show that a specified dependent variable (in this case  $g$ ) is a function of given independent variables; see Eqs. (B7) and (B9) in annex 2.
- $F\#2a$  = Eq. (5.5), which is a formula for numerical integration of precipitable water  $W$  in case there are only two pertinent data points available; this formula is based on linear interpolation.
- $F\#2b$  = Eq. (5.9), which is a formula for numerical integration of precipitable water  $W$  in case there are only two pertinent data points available; this formula is based on an assumed exponential interpolation.
- $F\#2$  = a general symbol employed to refer to both Eqs. (5.5) and (5.9); the one deemed to give the best fit may be picked.

F#3 = Eq. (5.12), which is a formula for numerical integration of precipitable water  $W$  in case three consecutive data points are used; this formula is based on Lagrangian interpolation.

F#4 = Eq. (5.18), which is a formula for numerical integration of precipitable water  $W$  in case four consecutive data points are used; this formula also is based on Lagrangian interpolation.

$g$  = acceleration of gravity, in  $\text{m}/\text{sec}^2$ ; see Eq.(B2).

$g_m$  = a properly weighted, integral mean value of the acceleration of gravity, in  $\text{m}/\text{sec}^2$ , taken over the interval from the surface to the top of the vertical air column whose precipitable water  $W$  is being computed; see Eqs. (3.3) and (4.3).

$g_0$  =  $9.8 \text{ m}^2/\text{sec}^2 \cdot \text{gpm}$  = constant = factor involved in the conversion from geometric altitude  $Z$ , in meters above sea level, to geopotential  $H$ , in geopotential meters (gpm); see Eq. (A4) in annex 1 and ref. 47 cited in annex 2.

$g_\phi$  = theoretical value of the acceleration of gravity, in  $\text{m}/\text{sec}^2$ , at sea level at the latitude  $\phi$  of the vertical air column whose precipitable water  $W$  is being computed; see Eq. (B1).

$G$  = coefficient in Eq. (5.18), defined by Eq. (5.21).

$H$  = geopotential, in geopotential meters (gpm).

$H_s$  = geopotential, in gpm, at the location of the base of the vertical air column whose precipitable water  $W$  is being computed.

$H_t$  = geopotential, in gpm, existing at the top of the vertical air column for which the precipitable water  $W$  is being computed.

$J$  = coefficient in Eq. (5.11).

$K$  = coefficient in Eq. (5.11); not to be confused with  $K_1$ .

$K_1$  = parameter defined by Eq. (B6) and used in Eq. (B3).

$\ln$  = natural logarithm.

$L$  = coefficient in Eq. (5.11); not to be confused with  $L_1$ .

$L_1$  = parameter defined by Eq. (B4) and used in Eq. (B3).

$m$  = mass of water vapor, in grams; see Eq. (A1).

$dm$  = differential mass, in grams, of water vapor contained within a horizontal slice of air whose vertical thickness is  $dZ$  and lateral boundary is formed by the (imaginary) vertical, cylindrical wall of the vertical air column whose precipitable water  $W$  is being calculated. That is,  $dm$  represents the differential mass of water vapor content of a disk of moist air cut horizontally from the above-mentioned vertical air column, where  $dZ$  is the thickness of the disk and  $A_c$  is its horizontal cross section area; see Eq. (A1).

$m.$  = symbol for meter (unit of length); and  $cm.$  = symbol for centimeter (unit of length).

$M$  = coefficient in Eq. (5.18)), defined by Eq. (5.22).

$M_a$  = apparent gram molecular weight of clean, dry atmospheric air = 28.9645 grams/mole.

$M_v$  = gram molecular weight of water vapor = 18.01534 grams/mole.

$M_v/M_a$  = constant = 0.62198 = ratio of molecular weight of water vapor to apparent molecular weight of clean, dry atmospheric air; dimensionless.

$N$  = total number of data points available in the range from the surface to the top of the vertical air column whose precipitable water  $W$  is being computed; thus  $N$  is determined by the number of significant levels and mandatory, isobaric levels, including the levels for the bottom (ground) and top of the column; see figs. 1 and 2.

$n$  = any integer determined from expressions like  $(N - 1)/3 = n$ ,  $(N - 1)/3 = (n + 1/3)$ , and  $(N - 1)/3 = (n + 2/3)$ ; see table 3, where  $N$  represents the total number of data points available for the numerical integration when computing  $W$ .

$P$  = barometric pressure; in the final integral expressions representing  $W$ , such as Eqs. (4.1), (4.2), (A22) and (A23), the pressure  $P$  is expressed in millibars (mb). The derivation of the relationship for  $W$  in annex A is initially based on the meter-gram-second system; hence, in Eqs. (A14) - (A18) it may be considered that  $P$  is expressed in terms of the unit (gram·m/m<sup>2</sup>·sec<sup>2</sup>); see Eqs. (A18) and (A19).

$P_s$  = barometric pressure, in millibars, at the location of the base of the vertical air column whose precipitable water  $W$  is being computed.

$P_t$  = barometric pressure, in millibars, at the upper limit of the vertical air column whose precipitable water  $W$  is being calculated.

- $q$  = specific humidity, in grams of water vapor per gram of moist air; dimensionless; see Eq. (A16).
- $Q$  = coefficient in Eq. (5.17).
- $R^*$  = universal gas constant =  $8.31432 \times 10^{-2}$  mb·m<sup>3</sup>/mole·°K; see Eq. (A7).
- $S$  = coefficient in Eq. (5.17).
- $T$  = absolute thermodynamic temperature, in degrees Kelvin (°K); see Eq. (A7).
- $U$  = coefficient in Eq. (5.17).
- $V$  = coefficient in Eq. (5.17).
- $W$  = precipitable water; in the final integral relationships representing  $W$ , such as Eqs. (3.1), (3.2), (4.1), (4.2), and the identical expressions derived in annex 1, the quantity  $W$  is expressed in terms of the unit hundredths of an inch of water; however, in the derivations the value of  $W$  is first given in terms of the meter units.
- $W_{s,5}$  = precipitable water (0.01 in.) in the vertical column of atmospheric air extending from the surface to the 500-mb level.
- $W_{s,7}$  = precipitable water, in 0.01 inch units, in the vertical air column extending from the surface to the 700-mb level.
- $W_{s,85}$  = precipitable water, in 0.01 inch units, in the vertical air column extending from the surface to the 850-mb level.
- $x$  = independent variable used as a general representation of the vertical coordinate (either  $H$  or  $P$ ) involved as the variable of integration in the formulas for  $W$ , such as Eqs. (3.1), (3.2), (4.1), and (4.2); see table 1 regarding the transformation from  $H$  or  $P$  to  $x$ .
- $x_s$  = the value of the independent variable (either  $H$  or  $P$ ) at the surface at the station where the radiosonde observation is made; hence,  $x_s$  stands for either  $H_s$  or  $P_s$ , depending upon whether  $H$  or  $P$  is the variable of integration involved in the formula for  $W$ .
- $x_t$  = the value of the independent variable  $x$  at the top of the vertical air column whose precipitable water  $W$  is being calculated; hence  $x_t$  stands for either  $H_t$  or  $P_t$ , depending upon whether  $H$  or  $P$  is the variable of integration involved in the formula for  $W$ .
- $x_a, x_b, x_c, \dots, x_{t-2}, x_{t-1}$   
= values of the independent variable  $x$  pertaining to certain data points intermediate to those relating to the surface ( $x_s$ ) and top ( $x_t$ ) of the vertical air column; these values serve as limits of the integrals representing "partial precipitable water." See Eq. (5.3).

$(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$

= four consecutive data points.

$y = f(x)$  = dependent variable (see fig. 1) used as a general representation of the integrand involved in the integral formulas for  $W$ , such as Eqs. (3.1), (3.2), (4.1), and (4.2); the integrand  $y$  is regarded as a function  $f(x)$  of the independent variable  $x$  as indicated in table 1.

$Z$  = geometric altitude (height above sea level), in meters; see Eq. (A1) and also Eq. (B3).

### Greek

$\phi$  = latitude.

$\lambda$  = longitude.

$\rho$  = density of moist air, in grams of moist air per cubic meter ( $\text{grams}/\text{m}^3$ ).

$\rho_v$  = absolute humidity, that is, vapor concentration or density of water vapor in the moist air mixture, in grams of water vapor per cubic meter ( $\text{grams}/\text{m}^3$ ); see Eq. (A7).

$\rho_w$  = density of liquid water, in  $\text{grams}/\text{m}^3$ , precipitated at the surface as rain which determines the depth of precipitable water  $W$ ; see Eqs. (A2) and (A6).

### Subscripts

$c$  refers to the cylindrical, vertical air column, particularly in connection with the cross-sectional area of the column. See  $A_c$ .

$s$  refers to the surface at the radiosonde station from which the instrument is launched.

$t$  refers to the top (upper limit) of the vertical air column whose precipitable water  $W$  is being determined.

$v$  refers to vapor of water substance.

$w$  refers to liquid water.

## 3. FORMULAS OF METHOD I FOR COMPUTING PRECIPITABLE WATER

According to the theory given in annex 1, the precipitable water may be computed with the following formula where the acceleration of gravity  $g$  is treated as a variable:

$$W = 8.36 \left( \frac{\text{m} \cdot ^\circ\text{K}}{\text{sec}^2 \cdot \text{mb} \cdot \text{gpm}} \right) (0.01 \text{ inch}) \int_{H_s}^{H_t} \left( \frac{e}{gT} \right) dH \quad (3.1)$$

where

$W$  = the precipitable water (expressed in hundredths of an inch) in the vertical column of atmospheric air whose base is at the geopotential of the surface  $H_s$  and whose top is at the geopotential  $H_t$ ; in which units of  $H_s$  and  $H_t$  must be consistent with those for  $H$ .

$e$  = aqueous vapor pressure (mb).

$T$  = absolute temperature of the air ( $^\circ\text{K}$ ).

$g$  = acceleration of gravity ( $\text{m}/\text{sec}^2$ ).

$H$  = geopotential (gpm).

Annex 2 gives equations permitting one to express the acceleration of gravity  $g$  as a function of the independent variable  $H$  (the geopotential to which the dependent variables  $e$ ,  $T$ , and  $g$  relate).

When values of the variables in the specified units are substituted under the integral sign of Eq. (3.1), the dimensions of the quantity  $(e/gT)dH$  exactly cancel the dimensions of the coefficient ( $\text{m} \cdot ^\circ\text{K}/\text{sec}^2 \cdot \text{mb} \cdot \text{gpm}$ ). After this cancellation, there remains in front of the integral the quantity (0.01 inch) which represents the dimension chosen for the precipitable water  $W$  in the present instance.

On the basis of the theory presented in annex 1, the precipitable water may be calculated with the following formula where one assumes that the acceleration of gravity  $g$  has the constant value  $9.8 \text{ m}/\text{sec}^2$ :

$$W = 0.853 \left( \frac{^{\circ}\text{K}}{\text{mb} \cdot \text{gpm}} \right) (0.01 \text{ inch}) \int_{H_s}^{H_t} \left( \frac{e}{T} \right) dH \quad (3.2)$$

approximately.

The definitions of the terms involved in Eq. (3.2) are the same as those of the corresponding terms in the text which immediately follows Eq. (3.1).

The treatment of dimensions and units in Eq. (3.2) is analogous to that described for Eq. (3.1).

Since Eq. (3.2) was derived from Eq. (3.1) under the assumption that  $g$  has a constant value equal to  $9.8 \text{ m/sec}^2$ , Eq. (3.2) is an approximation yielding different results from those yielded by Eq. (3.1) as the properly weighted mean value of  $g$  differs from the assumed constant  $9.8 \text{ m/sec}^2$ . The properly weighted mean value of  $g$ , which we here denote by  $g_m$ , is defined by the relationship

$$\frac{1}{g_m} \int_{H_s}^{H_t} \left( \frac{e}{T} \right) dH = \int_{H_s}^{H_t} \left( \frac{e}{gT} \right) dH \quad (3.3)$$

We can obtain a more accurate result for  $W$  than that yielded by Eq. (3.2) if we use either Eq. (3.1) or that equation revised by the substitution of  $g_m$  for  $g$  in the denominator.

Eq. (3.3) implies that  $g_m$  varies with the vertical distributions of  $(e/T)$  and  $(e/gT)$ , as well as with the latitude  $\phi$  and the lower and upper limits  $H_s$  and  $H_t$  of the integrals. Consequently, at many locations there tends to be a slight seasonal variation of  $g_m$  from summer to winter for constant values of  $H_s$  and variable values of the other quantities. Even the annual mean value of  $g_m$  may differ somewhat from  $9.8 \text{ m/sec}^2$  in many cases.

The conclusion is that if one chooses to employ Eq. (3.2) in practice it will be done at some slight cost in accuracy of the final result as compared with what could be determined on the basis of Eq. (3.1) or the modification of that relationship based on substituting for  $g$  the "best" value of  $g_m$  pertinent to the given locality. Estimates of the "best" seasonal and annual values of  $g_m$  may be calculated with the aid of climatological upper-air data. If it is ever decided to use such values, they should be applied consistently over the entire radiosonde network in order to realize the gain in accuracy.

## 4. FORMULAS OF METHOD II FOR COMPUTING PRECIPITABLE WATER

According to the theory presented in annex 1, the precipitable water may be calculated by means of the following relationship where the acceleration of gravity  $g$  is taken as a variable:

$$W = - 244.9 \left( \frac{m}{\text{sec}^2 \cdot \text{mb}} \right) (0.01 \text{ inch}) \int_{P_s}^{P_t} \frac{e}{g(P - 0.37802e)} dP \quad (4.1)$$

where

$W$  = the precipitable water (expressed in hundredths of an inch) pertinent to the vertical column of atmospheric air whose base is at the surface of the earth where the barometric pressure is  $P_s$  and whose top lies at the level where the barometric pressure is  $P_t$ ; in which units of  $P_s$  and  $P_t$  must be consistent with those of  $P$ .

$e$  = aqueous vapor pressure (mb).

$P$  = barometric pressure (mb).

$g$  = acceleration of gravity ( $m/\text{sec}^2$ ).

Annex 2 presents relationships which enable one to express the acceleration of gravity  $g$  as a function of the latitude  $\phi$  and the geopotential  $H$  which represents the level (height coordinate) corresponding to the general pressure  $P$  involved as the independent variable in Eq. (4.1). In order to apply those relationships, the user must have a table (or chart) representing the one-to-one correspondence between  $H$  and  $P$ ; that is, for every given value of  $P$  pertinent to any significant level or constant-pressure surface, the table (or chart) should indicate the corresponding geopotential  $H$  at the specified value of the pressure  $P$ . When performing a numerical integration of Eq. (4.1) for the case where the values of the independent variable, pressure  $P$ , are given for a series of significant levels and constant-pressure levels, the geopotentials  $H$  corresponding to those pressures must be "looked up" in the table (or chart). Then the resulting quantities representing the appropriate variable  $H$  can be used as arguments, together with the latitude  $\phi$ , in evaluating the relationships given in annex 2 to calculate the variable values of  $g$  required

in the denominator under the integral sign of Eq. (4.1).

In terms of the theory in annex 2, the precipitable water  $W$  in hundredths of an inch may be computed with the following equation where one assumes that the acceleration of gravity  $g$  has the constant value  $9.8 \text{ m/sec}^2$ :

$$W = -24.99 \left( \frac{1}{\text{mb}} \right) (0.01 \text{ inch}) \int_{P_s}^{P_t} \frac{e}{(P - 0.37802e)} dP \quad (4.2)$$

approximately,

where  $W$ ,  $P_s$ ,  $P_t$ ,  $P$  and  $e$  denote the quantities defined by the same symbols in the text which immediately follows Eq. (4.1). The millibar is used for the pressure unit.

Eq. (4.2) is an approximation because it was derived from the valid relationship Eq. (4.1) under the assumption that  $g$  has the constant value  $9.8 \text{ m/sec}^2$ .

Now let us define  $g_m$ , "the properly weighted mean value of  $g$ ," in accordance with the following relationship:

$$\left( \frac{1}{g_m} \right) \int_{P_s}^{P_t} \frac{e}{(P - ce)} dP = \int_{P_s}^{P_t} \frac{e}{g(P - ce)} dP \quad (4.3)$$

where

$c = 0.37802 = (1 - 0.62198)$  = constant in which 0.62198 represents the ratio of the molecular weight of water vapor to the apparent molecular weight of dry air.

It follows from Eq. (4.3) that  $g_m$  varies with the vertical distributions of  $e$ ,  $P$ , and  $g$ , as well as with the lower and upper limits  $P_s$  and  $P_t$  of the integrals. Since the ratio  $e/P$  is a function of the pressure  $P$ , the geographical coordinates  $(\phi, \lambda)$  and the time, as is known from climatological

data, the value of  $g_m$  will undergo a seasonal variation. When  $P_t = 500$  mb, the annual mean value of  $g_m$  is approximately equal to  $9.8 \text{ m/sec}^2$ , but not necessarily precisely equivalent to  $9.8 \text{ m/sec}^2$ .

By making use of climatological upper air data, it is possible to calculate from Eq. (4.3) what may be termed "seasonal best values of  $g_m$ " for each station or region characterized by uniformity of pressure regime. If such values of  $g_m$  are substituted for  $g$  in Eq. (4.1), the resultant calculated values of  $W$  are likely to be more accurate in general than those computed on the basis of Eq. (4.2).

5. NUMERICAL INTEGRATION BASED ON TWO, THREE  
AND FOUR DATA POINTS OF NONLINEAR FUNCTIONS

The relationships expressed by Eqs. (3.1), (3.2), (4.1) and (4.2) require numerical integration of nonlinear, empirically determined functions. These may be represented symbolically in the general form

$$y = f(x) \quad (5.1)$$

where

$x$  = the independent variable of the data point;

$y$  = the dependent variable of the data point.

On the basis of the observations we shall have available a number of data points with coordinates such as  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ , etc. These are illustrated graphically in fig. 1 where it is indicated that, in general, the spacings between the abscissas of consecutive data points are unequal. On the basis of the foregoing general expression for the empirically determined functions, the relationships representing the precipitable water  $W$  may be transformed from Eqs. (3.1), (3.2), (4.1) and (4.2) to

$$W = C \int_{x_s}^{x_t} y \, dx = C \int_{x_s}^{x_t} f(x) \, dx \quad (5.2)$$

where

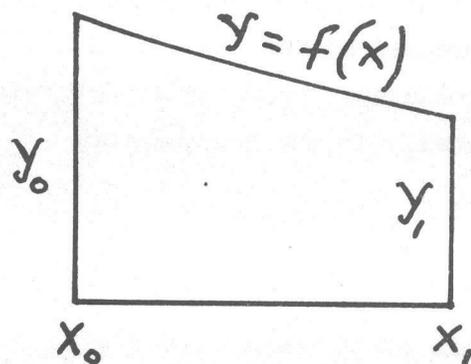
$C$  = a suitable dimensional constant representing the coefficient of the integral in whichever of Eqs. (3.1), (3.2), (4.1) or (4.2) is under consideration.

$x_s$  = the value of the lower limit of whichever of those equations is being evaluated; (this limit refers to the surface or bottom of the vertical column).

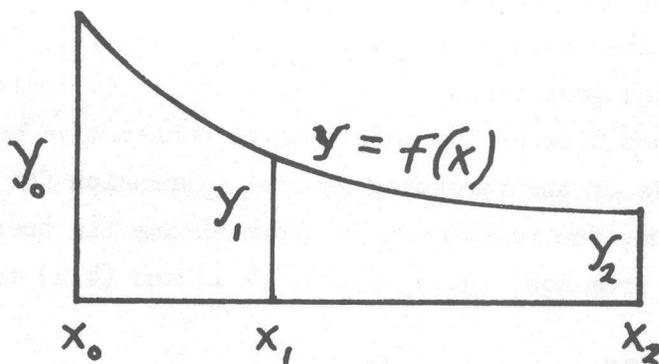
$x_t$  = the value of the upper limit of whichever of those equations is being evaluated; (this limit refers to the top of the vertical column).

$y = f(x)$  = the value of the integrand shown under the integral sign, depending on whichever of the specified equations is involved.

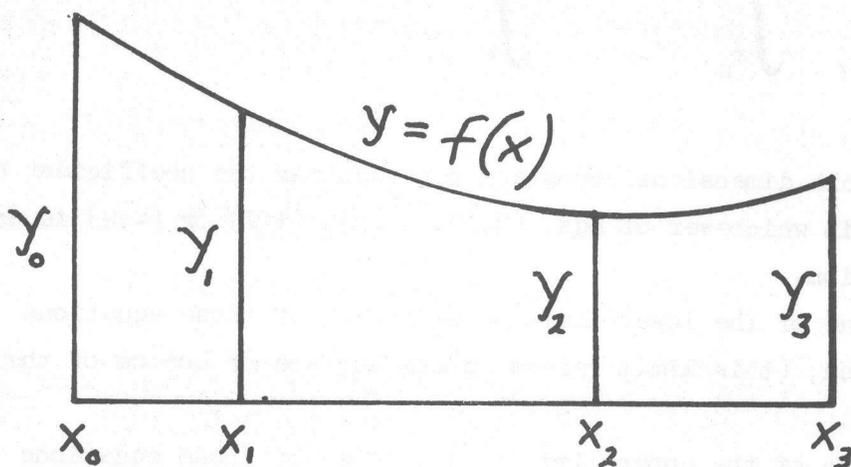
Examples of curves indicating empirical functions,  $y = f(x)$ , based on 2-, 3-, and 4-consecutive data points.



2  
points



3  
points



4  
points

Fig. 1

$x$  = the variable with respect to which the integration is performed; it represents the independent variable (H or P).

The following table indicates the transformation of coordinates and limits, depending upon the equation involved.

Table 1

Transformation of Coordinates and Limits

Equation No.	Independent Variable $x$	Integrand $y$	Lower Limit $x_s$	Upper Limit $x_t$
(3.1)	H	$\frac{e}{gT}$	$H_s$	$H_t$
(3.2)	H	$\frac{e}{T}$	$H_s$	$H_t$
(4.1)	P	$\frac{e}{g(P - 0.37802e)}$	$P_s$	$P_t$
(4.2)	P	$\frac{e}{(P - 0.37802e)}$	$P_s$	$P_t$

If one uses values derived from both significant levels and from mandatory, constant-pressure levels as a source for the data points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ , etc., there will usually be a considerable number of data points in the range from the lower to the upper limit ( $x_s$  to  $x_t$ ), where  $x_t$  generally refers to the 500-mb level for the top of the vertical column of atmospheric air whose precipitable water  $W$  is being determined. One may fit a continuous polynomial (such as a power series) or other suitable empirical function to the data points in order to ascertain an analytical expression for the function  $y = f(x)$  pertinent to each sounding, but this would be exceedingly cumbersome if the total number  $N$  of available data points in the range from  $x_s$  to  $x_t$  is relatively large, especially since the spacings between the abscissas  $x$  is generally irregular, as illustrated in fig. 1. The vast majority of radiosonde observations will usually provide at least three data points in the range from  $x_s$  to  $x_t$ , and many of the observations will provide at least four data points in this range. Rarely will there be just two data points;

that is, one pertinent to the surface (at abscissa  $x_g$ ) and the other pertinent to the 500-mb level when this is taken for the top of the vertical column (at abscissa  $x_t$ ), with no other available significant levels or mandatory, constant-pressure levels intermediate to  $x_g$  and  $x_t$ .

Owing to the fact that it is feasible to fit a polynomial passing through either three or four consecutive data points without obtaining unduly cumbersome expressions, we shall indicate how the desired numerical integrations may be performed on this basis. Then, in order to evaluate the integral Eq. (5.2) for cases where the total number  $N$  of available data points in the range from  $x_g$  to  $x_t$  is greater than four (4), we shall divide the range from  $x_g$  to  $x_t$  into segments, each consisting of either three or four consecutive data points. On this basis the technique of numerical integration derived for the case of three and four consecutive points, respectively, may be applied to each of the various segments, as will be appropriate, depending upon the number of data points embraced in the individual segment. See fig. 2.

It will be clear that the numerical integration of an equation like Eq. (5.2) between lower and upper limits which relate to levels lying entirely in the free air will yield what we have termed "partial precipitable water" as explained in the eighth paragraph of sec. 1. Thus, in cases having a total number of data points  $N$  greater than four (4) in the range from  $x_g$  to  $x_t$ , we shall calculate the "partial precipitable water" pertaining to segments involving three or four consecutive data points; and consequently it follows that the "precipitable water"  $W$  relating to the total range from  $x_g$  to  $x_t$  (such as from the surface to the 500-mb level) will be computed by summing up the "partial precipitable water" contributions yielded by the various consecutive segments into which the entire range from  $x_g$  to  $x_t$  is divided. This is exemplified by the following relationship:

$$\begin{aligned}
 W = C \int_{x_s}^{x_t} f(x)dx &= C \int_{x_s}^{x_a} f(x)dx + C \int_{x_a}^{x_b} f(x)dx + C \int_{x_b}^{x_c} f(x)dx \\
 &+ \dots + C \int_{x_{t-2}}^{x_{t-1}} f(x)dx + C \int_{x_{t-1}}^{x_t} f(x)dx
 \end{aligned} \tag{5.3}$$

where  $x_a, x_b, x_c, \dots, x_{t-2}, x_{t-1}$  represent limits in a sequence intermediate to  $x_s$  and  $x_t$ .

In the material that follows we shall present formulas recommended for use in the numerical integrations implied by Eq. (5.3) where, in general, either three or four consecutive data points will be embraced between the lower and upper limits of the integrals on the right-hand side of the equation. We shall also present numerical-integration formulas for the rare case in which there exist only two available data points in the total range from  $x_s$  to  $x_t$ . For the sake of brevity we will employ the following identification symbols with reference to these formulas:

Table 2

Identification Symbols Relating to Numerical Integration  
Formulas Pertinent to the Cases of Two, Three and  
Four Consecutive Data Points

Number of consecutive data points used in the numerical integration between the lower and upper limits involved	Identification symbols employed in referring to the formulas
2	F#2 (or F#2a, F#2b)
3	F#3
4	F#4

Detailed specifications are given later regarding formulas F#2a, F#2b, F#3 and F#4.

Fig. 2 shows the method which we recommend for dividing the range from  $x_s$  to  $x_t$  into segments, depending upon the total number  $N$  of data points within that range. The identification symbols of the numerical-integration formulas pertinent to the segments embracing two, three or four consecutive data points are indicated beneath the segments in terms of the notation introduced above: F#2, F#3 and F#4, as appropriate. It will be noted that in cases where  $N$  is 5 or more, we recommend that first priority be given to F#4 and second priority to F#3; that is, a maximum number of segments involving four consecutive data points should be taken, and the remaining segments should involve three consecutive data points, thus obviating the use of F#2 for cases where  $N$  is 5 or more.

The following table indicates general and specific rules for the application of formulas F#2, F#3 and F#4, based on dividing the range  $x_s$  to  $x_t$  into segments as illustrated by fig. 2.

Integration formula (whether #2, #3, or #4) to be used in computing  $W$  for the vertical air column or its segments, depending on number of data points in range from bottom to top of column.

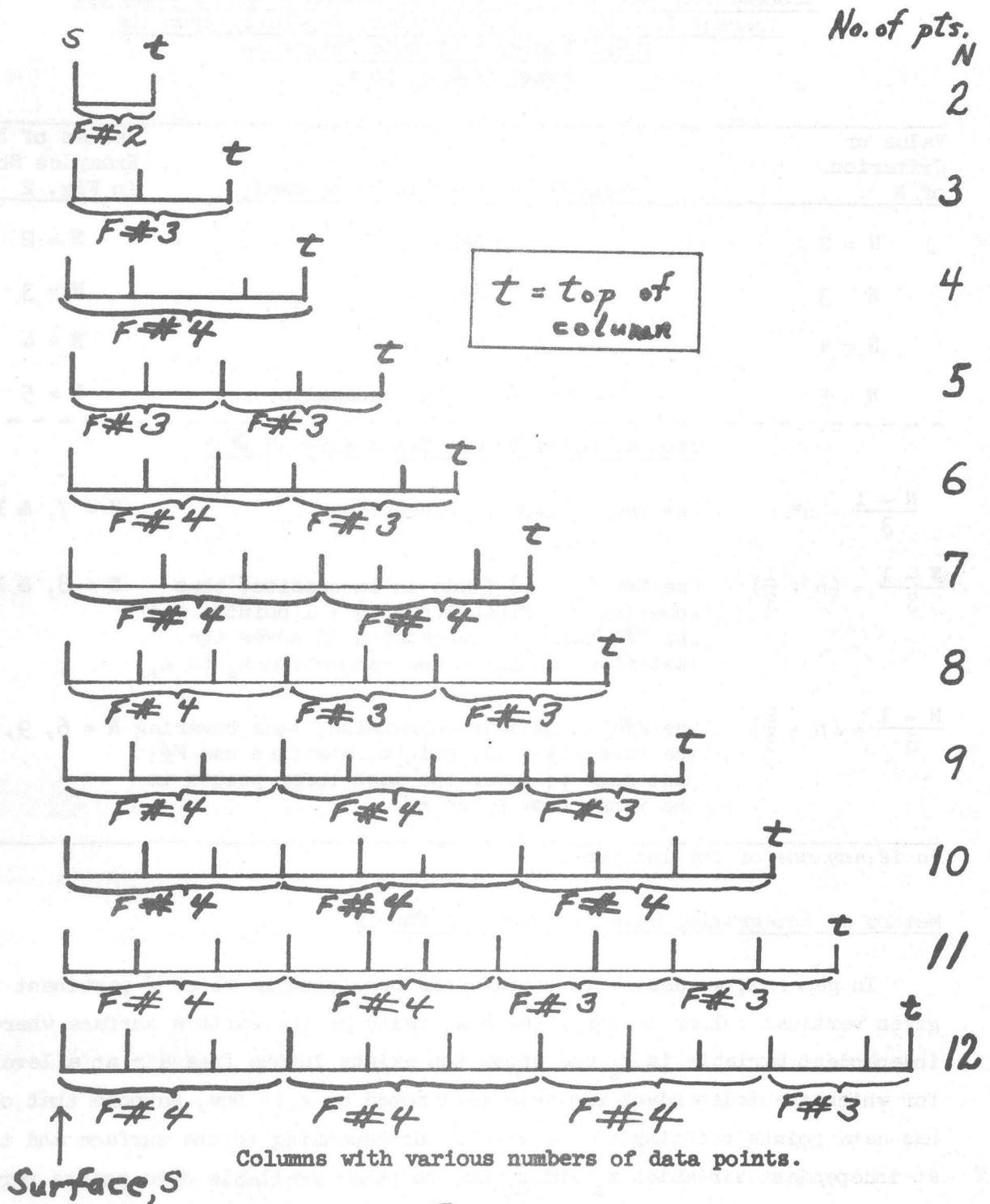


Fig. 2

Table 3

Numerical-Integration Formulas F#2, F#3 and F#4 To Be Used for  
Integration over the Vertical-Air-Column Range or Segments  
Thereof Into Which It Is Divided, Depending upon the  
Total Number N of Data Points in  
Range from  $x_s$  to  $x_t$

Value or Criterion of N	Formula or Formulas To Be Used	Values of N in Examples Shown in Fig. 2
N = 2	F#2	N = 2
N = 3	F#3	N = 3
N = 4	F#4	N = 4
N = 5	Use F#3 twice in succession	N = 5

-----  
General Rules To Fit Cases Where  $N > 5$

$\frac{N-1}{3} = n^*$	Use F#4 n times in succession	N = 7, & 10
$\frac{N-1}{3} = (n + \frac{1}{3})$	Use F#4 (n - 1) times in succession, thus covering the first $3(n-1) + 1$ points, and use F#3 twice in succession to cover the last five points in the range from $x_s$ to $x_t$	N = 8, & 11
$\frac{N-1}{3} = (n + \frac{2}{3})$	Use F#4 n times in succession, thus covering the first $(3n+1)$ points, and then use F#3 just once to cover the last three points in the range from $x_s$ to $x_t$	N = 6, 9, & 12

\*n is any one of the integers.

Method of Integration Based on Two Data Points

In general, we desire to compute the precipitable water W pertinent to a given vertical column of air whose base rests on the earth's surface where the independent variable is  $x_s$  and whose top exists in the free air at a level for which the independent variable is denoted by  $x_t$ . Now, suppose that one has data points relating to the levels corresponding to the surface and top at independent variables  $x_s$  and  $x_t$  but no other available data points pertaining to intermediate levels.

Since there are only two data points in this case it is impossible to determine the form of the function  $y = f(x)$  for the range from  $x_s$  to  $x_t$ , whether linear or nonlinear.

In what follows we shall present simple procedures for obtaining the result of the integration of  $f(x)dx$  between the limits  $x_s$  and  $x_t$  on the basis of either of two specified assumptions with regard to the form of the function  $f(x)$ . It is recommended that these methods be employed exclusively in the case where only the two data points for the levels at  $x_s$  and  $x_t$  are available, but no others for levels between them.

#### Method Based on Linear Form of $f(x)$ , Given Two Data Points

Let us assume that within the range from  $x_s$  to  $x_t$  the function under consideration is represented by the linear relationship

$$y = f(x) = y_s + \frac{(y_t - y_s)}{(x_t - x_s)}(x - x_s) \quad (5.4)$$

When this is substituted in Eq. (5.2), it yields

$$W = \frac{C}{2}(x_t - x_s)(y_s + y_t) \quad \text{denoted F\#2a} \quad (5.5)$$

This corresponds to the so-called Trapezoidal Rule cited in many books on numerical analysis (see, for example, Ref. 88).

For brevity we designate the formula given by Eq. (5.5) as F#2a.

The use of F#2a should be restricted to cases where  $(x_t - x_s)$  is relatively small.

#### Method Based on Exponential Form of $f(x)$ , Given Two Data Points

When one restricts attention to situations where the vertical distance between the limits  $x_s$  and  $x_t$  is not very great, climatological data suggest that the vertical distributions of the integrands in Eqs. (3.1), (3.2), (4.1) and (4.2) may be closely approximated by exponential functions of the vertical coordinate involved as the independent variable under the integral. Using

this as justification, we shall assume for present purposes that within the range from  $x_s$  to  $x_t$

$$y = y_s e^{b(x - x_s)} \quad (5.6)$$

and

$$y_t = y_s e^{b(x_t - x_s)} \quad (5.7)$$

where

$e$  = base of natural logarithms;

$b$  = constant, defined by Eq. (5.7), for given observational data relating to  $x_s, x_t, y_s$  and  $y_t$ .

From Eq. (5.7) we obtain

$$b = \frac{(\ln y_t - \ln y_s)}{(x_t - x_s)} \quad (5.8)$$

When Eq. (5.8) is substituted in Eq. (5.6), and the latter is then substituted in Eq. (5.2), this yields, after a little reduction,

$$W = C \frac{(x_t - x_s)(y_t - y_s)}{(\ln y_t - \ln y_s)} \quad \text{denoted F\#2b} \quad (5.9)$$

For brevity we shall designate the formula expressed by Eq. (5.9) as F#2b. Collectively, we shall designate the pair F#2a and F#2b by the notation F#2.

Under conditions where the only available data points are  $(x_s, y_s)$  and  $(x_t, y_t)$ , with no others given between them, while the difference  $(x_t - x_s)$  is not excessively large in absolute value, preference should be given to the use of F#2b over the use of F#2a.

#### Method of Integration Based on Three Data Points

As may be seen from fig. 2, there will be numerous cases where one wishes to perform the numerical integration of  $f(x)dx$  over a range for which there will be three consecutive data points. For the sake of simplicity we shall designate these points by means of the notation:  $(x_0, y_0)$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$ , where the first refers to the lowermost and the last refers to the uppermost

on the basis of geopotential.

We assume that  $f(x)$  in the range from  $x_0$  to  $x_2$  is represented by a polynomial of the form

$$\begin{aligned}
 y = f(x) = & \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 \\
 & + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 \\
 & + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2
 \end{aligned} \tag{5.10}$$

It is readily verified that the curve represented by Eq. (5.10) automatically passes through the three specified consecutive data points. Application of this relationship is tantamount to the use of Lagrange's formula of interpolation (see Ref. 88).

Now, by expanding the right-hand side of Eq. (5.10) and collecting the terms which relate to the common powers of  $x$ , respectively, we may re-write that equation in the form of a quadratic:

$$y = f(x) = Jx^2 + Kx + L \tag{5.11}$$

where  $J$ ,  $K$  and  $L$  are functions of the coordinates of the three consecutive data points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ . We do not take the trouble to write the expressions for  $J$ ,  $K$  and  $L$  here because they are easily derived by expansion of Eq. (5.10), after which it is equated to Eq. (5.11).

Next we substitute Eq. (5.11) in Eq. (5.2) where, this time, we make use of the expressions for  $J$ ,  $K$  and  $L$ . Concurrently, we replace the lower limit  $x_s$  in the integral by  $x_0$  and the upper limit  $x_t$  by  $x_2$  because in the special case under consideration the lowest data point  $(x_0, y_0)$  may be regarded as a substitution for lower limiting data point  $(x_s, y_s)$  of the original integral, while the highest data point  $(x_2, y_2)$  may be regarded as a substitution for the upper limiting data point  $(x_t, y_t)$  of the original integral.

On performing the integration we obtain the solution in the following

form, valid only in the case of three data points:

$$W = A\{(x_2)^3 - (x_0)^3\} - B\{(x_2)^2 - (x_0)^2\} + D\{x_2 - x_0\} \quad \text{denoted as F\#3} \quad (5.12)$$

where the values of the coordinates of the three consecutive data points  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$  govern the coefficients A, B and D of the foregoing relationship, as indicated by the following three equations.

Letting C = the appropriate coefficient of the integrals in Eqs. (3.1), (3.2), (4.1) and (4.2), depending upon which one is being evaluated, we define the coefficients A, B and D in Eq. (5.12) by means of the relationships

$$A = \frac{C}{3} \left[ \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} \right] \quad (5.13)$$

$$B = \frac{C}{2} \left[ \frac{y_0(x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1(x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2(x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} \right] \quad (5.14)$$

$$D = C \left[ \frac{y_0(x_1 x_2)}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1(x_0 x_2)}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2(x_0 x_1)}{(x_2 - x_0)(x_2 - x_1)} \right] \quad (5.15)$$

After Eqs. (5.13), (5.14) and (5.15) are substituted in Eq. (5.12), it may be used to compute the precipitable water  $W$  or "partial precipitable water" in connection with vertical air columns for which three consecutive data points are taken (see, for example, the middle diagram in fig. 1 and the segments in fig. 2 with an underlying label F#3).

#### Method of Integration Based on Four Data Points

Numerical integration of  $f(x)dx$  over a range covering four consecutive data points will be required according to the scheme outlined in fig. 2 (see the segments with the underlying label F#4). Consistent with the bottom diagram in fig. 1, we shall denote these four available points by the notation:  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ , where the value of the independent variable  $x$  increases from left to right.

We assume that  $f(x)$  in the range from  $x_0$  to  $x_3$  is represented by a polynomial having the form

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3 \quad (5.16)$$

The function expressed by Eq. (5.16) is so designed that the curve which it represents will automatically pass through the four specified consecutive data points. It exemplifies Lagrange's formula of interpolation for this condition. (See Ref. 88.)

When we expand the right-hand side of Eq. (5.16) and collect the terms which relate to the common powers of  $x$ , respectively, we find that we may re-write that equation in the form of a cubic equation:

$$y = f(x) = Qx^3 + Sx^2 + Ux + V \quad (5.17)$$

where  $Q$ ,  $S$ ,  $U$  and  $V$  are functions of the coordinates of the four consecutive data points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ . Expressions for  $Q$ ,  $S$ ,  $U$  and  $V$  will not be written here because one can easily derive them by expansion of Eq. (5.16).

On substituting Eq. (5.17) in Eq. (5.2) and making use of the expressions for  $Q$ ,  $S$ ,  $U$  and  $V$ , we may perform the indicated integration and thus obtain the following relationship pertinent only for the case of four consecutive data points:

$$W = E\{(x_3)^4 - (x_0)^4\} - F\{(x_3)^3 - (x_0)^3\} \\ + G\{(x_3)^2 - (x_0)^2\} - M\{(x_3) - (x_0)\} \quad \text{denoted by } F\#4 \quad (5.18)$$

where the coefficients  $E$ ,  $F$ ,  $G$  and  $M$  depend upon the values of the coordinates of the four consecutive data points  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ , as indicated by the four equations which follow. The quantity  $C$  in these expressions represents the appropriate coefficient in Eqs. (3.1), (3.2), (4.1) and (4.2), depending upon which of them is being applied. In addition, we have replaced the lower limit  $x_s$  in these integrals by  $x_0$ , and the upper limit  $x_t$  by  $x_3$  for obvious reasons.

$$\begin{aligned}
 E = \frac{C}{4} & \left[ \frac{y_0}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \right. \\
 & + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
 & + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\
 & \left. + \frac{y_3}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \right] \quad (5.19)
 \end{aligned}$$

$$\begin{aligned}
 F = \frac{C}{3} & \left[ \frac{y_0(x_1 + x_2 + x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \right. \\
 & + \frac{y_1(x_0 + x_2 + x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
 & + \frac{y_2(x_0 + x_1 + x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\
 & \left. + \frac{y_3(x_0 + x_1 + x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \right] \quad (5.20)
 \end{aligned}$$

$$\begin{aligned}
 G = \frac{C}{2} & \left[ \frac{y_0(x_1x_2 + x_1x_3 + x_2x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \right. \\
 & + \frac{y_1(x_0x_2 + x_0x_3 + x_2x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
 & + \frac{y_2(x_0x_1 + x_0x_3 + x_1x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\
 & \left. + \frac{y_3(x_0x_1 + x_0x_2 + x_1x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \right] \quad (5.21)
 \end{aligned}$$

$$\begin{aligned}
 M = C & \left[ \frac{y_0(x_1 x_2 x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \right. \\
 & + \frac{y_1(x_0 x_2 x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
 & + \frac{y_2(x_0 x_1 x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\
 & \left. + \frac{y_3(x_0 x_1 x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \right] \quad (5.22)
 \end{aligned}$$

When Eqs. (5.19), (5.20), (5.21) and (5.22) are substituted in Eq. (5.18), it may be applied to calculate the precipitable water  $W$  or the "partial precipitable water" pertinent to vertical air columns for which four consecutive data points are used (see, for example, the lowest diagram in fig. 1 and the segments in fig. 2 having an underlying label  $F\#4$ ).

## 6. SUMMARY OF INTEGRATION PROCEDURES DEPENDING UPON TOTAL NUMBER OF CONSECUTIVE DATA POINTS AVAILABLE

Now let  $N$  denote the total number of data points available in the range from the surface to the top of the vertical column of atmospheric air whose precipitable water  $W$  is to be calculated. One should use all existing significant levels and mandatory, constant-pressure levels within that range in order to make  $N$  a maximum. The lowest level relates to the surface, and the independent variable (either geopotential  $H$  or pressure  $P$ ) pertinent to the lowest level is designated by  $x_s$  in general; whereas the highest level (top) of the vertical column of air under consideration (for example, the 500-mb level) has its independent variable designated by  $x_t$  in general. Thus,  $N$  represents the actual number of data points available for the calculations in the range from  $x_s$  to  $x_t$ , including these limits.

The theoretically derived expressions for precipitable water  $W$  are given by Eqs. (3.1), (3.2), (4.1) and (4.2), depending upon whether  $H$  or  $P$  is to be taken as the independent variable, and whether the acceleration of gravity  $g$  is to be treated as a variable of the problem or is to be regarded as a constant (in particular having the mean value  $9.8 \text{ m/sec}^2$ ).

In order to generalize the relationships we have transformed Eqs. (3.1), (3.2), (4.1) and (4.2) to the universal form of integral expression shown in Eq. (5.2), where  $f(x)$  denotes the integrand of the former equations, depending upon which one is being applied, and  $x$  denotes the independent variable (either  $H$  or  $P$ ). Symbol  $C$  is employed to represent the coefficient of Eq. (3.1), (3.2), (4.1) or (4.2), depending upon which one is being used.

The following procedures are recommended with regard to the fitting of curves to the available data points and the actual method of integration of the relevant formulas:

### (1) Case where $N = 2$

When  $N = 2$ , that is, when the only available data points are  $(x_s, y_s)$  and  $(x_t, y_t)$  relating to the surface and top, respectively, of the vertical column of air whose precipitable water  $W$  is being computed, one should calculate  $W$  by means of either Eq. (5.5), denoted F#2a, or Eq. (5.9), denoted F#2b. The former

(F#2a), being based upon linear interpolation, should be employed only if the difference between the limits of the integral ( $x_t - x_s$ ) is relatively small; while the latter (F#2b), being based on an exponential function, is considered preferable for use when ( $x_t - x_s$ ) cannot be regarded as relatively small. We suggest tentatively that 200 gpm, or the equivalent in pressure units, be taken as the upper limit of relatively small in this context.

(2) Case where  $N > 2$

Curve fitting by means of Lagrangian interpolation is the basis of this method. When  $N$  is greater than two (2), it is recommended that the range from  $x_s$  to  $x_t$  be divided into segments, if necessary, in order that the segments encompass either three or four consecutive data points in accordance with the scheme illustrated in fig. 2 and specified in table 3. Then Eq. (5.12), denoted F#3, should be employed for the numerical integration relating to any vertical column or segment thereof characterized by three consecutive data points. Similarly Eq. (5.18), denoted by F#4, should be used for the numerical integration pertaining to any vertical column or segment thereof involving just four consecutive data points. When the vertical air column is divided into segments as depicted in fig. 2 for cases where  $N > 4$ , then the precipitable water  $W$  relating to the entire vertical air column will be obtained by adding the separate contributions of "partial precipitable water" from the various segments into which the vertical air column as a whole was divided. Justification for this process is indicated by Eq. (5.3).

## 7. CONCLUSIONS

(1) When computing precipitable water  $W$ , one must evaluate an integral by a suitable numerical procedure. Under the integral sign there is a quantity, the integrand, such as empirically-determined specific humidity or absolute humidity, which generally varies non-linearly with respect to the variable of integration (such as pressure or height). Under many conditions, the empirical quantity is ascertained for unequal spacings of the variable of integration. The foregoing facts justify the use of the Lagrangian formula as exemplified by Eqs. (5.10) and (5.16) to facilitate the numerical integration. Explicit expressions for the coefficients required when using this formula in numerical integration can be specified as shown for the cases where either three or four consecutive data points are given. See the groups of relationships: Eqs. (5.10) - (5.15) and (5.16) - (5.22)

(2) In terms of the notation introduced in the second section of this paper, several different combinations of quantities may be involved in the integration as shown by the following table:

Case No.	Integrand	Variable of Integration
1	$e/gT$	H
2	$e/T$	H
3	$e/g(P - 0.37802e)$	P
4	$e/(P - 0.37802e)$	P

Case Nos. 2 and 4 entail the assumption that the acceleration of gravity  $g$  is constant. Although it is conventional to take  $9.8 \text{ m/sec}^2$  as the assumed constant value of  $g$ , it is possible by the method indicated in the paper to determine a more precise mean value for each region covering a delimited area bounded by specified circles of latitude. Procedures given for taking account of variable  $g$  are feasible to use with high-speed electronic computers.

(3) The fact is often overlooked that, when radiosonde observations are employed as the basis of precipitable water calculations, the results refer, strictly speaking, to a sloping filament following the trajectory of the instrument during the sounding, rather than a vertical column as assumed in the definition of  $W$ .

## REFERENCES

1. Adem, J., "A Parametric Method for Computing the Mean Water Budget of the Atmosphere," Tellus, Vol. 20, 1968, pp. 621-632.
2. Alavi-Nejad, M., "Appareil pour l'étude de la variation diurne de l'épaisseur d'eau condensable de l'atmosphère," Comptes Rendus, Vol. 254, 26 Feb 1962, pp. 1662-1664.
3. \_\_\_\_\_, "Amplitude de la variation diurne de l'épaisseur d'eau condensable de l'atmosphère," Comptes Rendus, Vol. 256, 21 Jan 1963, pp. 982-984.
4. Ananthakrishnan, R., M. M. Selvam and R. Chellappa, "Seasonal Variation of Precipitable Water Vapour in the Atmosphere over India," Indian Journal of Meteorology and Geophysics, Vol. 16(3), July 1965, pp. 371-384.
5. Bannon, J. K., A. G. Matthewman and R. Murray, "The Flux of Water-Vapour Due to the Mean Winds and the Convergence of This Flux over the Northern Hemisphere in January and July," Quarterly Journal of the Royal Meteorological Society, Vol. 88, July 1962, p. 339.
6. \_\_\_\_\_ and L. P. Steele, "Average Water-Vapour Content of the Air," Great Britain Meteorological Office, Air Ministry, London, Geophysical Memoirs, No. 102, Vol. XIII, 1960, 38 pp.
7. Barnes, A. A., Jr., "Atmospheric Water Vapor Divergence Measurements and Applications," in "Humidity and Moisture Measurement and Control in Science and Industry," (A. Wexler, Editor-in-Chief), Vol. 2, "Applications," (E. J. Amdur, Editor), Based on papers presented at the 1963 International Symposium on Humidity and Moisture, Washington, D.C., New York: Reinhold Publishing Corporation, 1965, pp. 513-522.
8. Battan, L. J., "Silver-Iodide Seeding and Rainfall from Convective Clouds," Journal of Applied Meteorology, Vol. 5, 1966, pp. 669-683.
9. Benton, G. S., "Atmospheric Studies of Large Scale Transfer Fields over the North American Continent," Department of Civil Engineering, The Johns Hopkins University, Baltimore, Maryland, Final Report, Vol. I, on Contract No. AF 19(122)-365, Geophysics Research Directorate, Air Force Cambridge Research Center, Bedford, Mass., 1954, 247 pp.
10. \_\_\_\_\_, "Presentation of Meteorological Data for the North American Sector for the Year 1949," Department of Civil Engineering, The Johns Hopkins University, Baltimore, Maryland, Final Report, Vol. II, on Contract No. AF 19(122)-365, Geophysics Research Directorate, Air Force Cambridge Research Center, Bedford, Mass., 1954, 182 pp.
11. \_\_\_\_\_, R. T. Blackburn and V. O. Snead, "The Role of the Atmosphere in the Hydrologic Cycle," Transactions of the American Geophysical Union, Vol. 31, 1950, pp. 61-73.
12. \_\_\_\_\_ and M. A. Estoque, "Water Vapor Transfer over the North American Continent," Journal of Meteorology, Vol. 11, 1954, pp. 462-477.

13. \_\_\_\_\_, \_\_\_\_\_ and J. Dominitz, "An Evaluation of the Water Vapor Balance of the North American Continent," The Johns Hopkins University, Baltimore, Maryland, Scientific Report No. 1, Contract No. AF 19(122)-365, Geophysics Research Directorate, Air Force Cambridge Research Center, Bedford, Mass., 1953, 101 pp.
14. \_\_\_\_\_, R. G. Fleagle, D. F. Leipper, R. B. Montgomery, N. Rakestraw, W. S. Richardson, H. Riehl and J. Snodgrass, "Interaction Between the Atmosphere and the Oceans," Bulletin of the American Meteorological Society, Vol. 44, 1963, pp. 4-17.
15. Bock, P., H. M. Frazier and J. G. Welsh, "Moisture Flux over North America. Part II. Analyses of Monthly and Longer-term Means of Flux Divergence of Moisture over North America, May 1958-April 1963," The Travelers Research Center, Inc., Hartford, Conn., Final Report 7477-244 on Contract Cwb-11313, Environmental Science Services Administration, Weather Bureau, Silver Spring, Md., 1966, 99 pp.
16. Bolsenga, S. J., "The Relationship Between Total Atmospheric Water Vapor and Surface Dew-Point on a Mean Daily and Hourly Basis," Journal of Applied Meteorology, Vol. 4, 1965, pp. 430-432.
17. Bradbury, D. L., "Moisture Analysis and Water Budget in Three Different Types of Storms," Journal of Meteorology, Vol. 14, December 1957, pp. 559-565.
18. \_\_\_\_\_, "Moisture Analysis and Water Budget in Some Nondeveloping Cases," Geophysica, Vol. 6, 1958, pp. 179-187.
19. Byers, H. R., "Significance of Different Vertical Distributions of Water Vapor in Arid and Humid Regions," Journal of the Meteorological Society of Japan, 75th Anniversary Volume, November 1957, pp. 330-335.
20. Crisi, A. R., "Hemispheric Water Vapor Balance During 1958," in "Humidity and Moisture Measurement and Control in Science and Industry," (A. Wexler, Editor-in-Chief), Vol. 2, "Applications," (E. J. Amdur, Editor), Based on papers presented at the 1963 International Symposium on Humidity and Moisture, Washington, D.C., New York: Reinhold Publishing Corporation, 1965, pp. 502-512.
21. Ferguson, H. L. and A. D. J. O'Neill, "Atmospheric Water Balance over an Area of 30,000 km<sup>2</sup>," Meteorological Branch, Department of Transport, Toronto, Canadian Meteorological Research Reports, I.H.D. Project R-L-7 CAN 27, International Field Year on the Great Lakes, Feasibility Study SP-1, CMRR 7/68, September 20, 1968, 17 pp.
22. Flohn, H. and H. Oeckel, "Water Vapour Flux During the Summer Rains over Japan and Korea," The Geophysical Magazine (Tokyo), vol. 27, 1956, pp. 527-532.
23. Foster, N. B. and L. W. Foskett, "A Spectrophotometer for the Determination of the Water Vapor in a Vertical Column of the Atmosphere," Journal of the Optical Society of America, Vol. 35, 1945, pp. 601-610.

24. Foster, N. B., D. T. Volz and L. W. Foskett, "A Spectral Hygrometer for Measuring Total Precipitable Water" in "Humidity and Moisture Measurement and Control in Science and Industry," (Arnold Wexler, Editor-in-Chief), Vol. 1, "Principles and Methods of Measuring Humidity in Gases," (Robert E. Ruskin, Editor), Based on papers presented at the 1963 International Symposium on Humidity and Moisture, Washington, D.C., New York: Reinhold Publishing Corporation, 1965, pp. 455-464.
25. Fowle, F. E., "The Spectroscopic Determination of Aqueous Vapor," Astro-physical Journal, Vol. 35, 1912, p. 149.
26. Gates, D. M., "Stratospheric Moisture Measurements Using Infrared Spectroscopy" in "Humidity and Moisture Measurement and Control in Science and Industry," (Arnold Wexler, Editor-in-Chief), Vol. 2, "Applications," (Elias J. Amdur, Editor), Based on papers presented at the 1963 International Symposium on Humidity and Moisture, Washington, D.C., New York: Reinhold Publishing Corporation, 1965, pp. 446-449.
27. Goldschmidt, J., "Die Bestimmung des Niederschlagswassers der Atmosphäre aus Radiosondenaufstiegen und optischen Messungen," Zeitschrift für Meteorologie, Band 5, 1951, p. 152.
28. Gutnick, M., "An Estimate of Precipitable Water Along High-Altitude Ray Paths," Geophysics Research Directorate, Air Force Cambridge Research Laboratories, Bedford, Mass., Air Force Surveys in Geophysics No. 120, AFCRC-TN-60-251, March 1960, 26 pp.
29. \_\_\_\_\_, "Mean Annual Mid-Latitude Moisture Profiles to 31 km," Geophysics Research Directorate, Air Force Cambridge Research Laboratories, Bedford, Mass., Air Force Surveys in Geophysics No. 147, AFCRL-62-681, July 1962, 30 pp.
30. Haltiner, G. J. and F. L. Martin, "Dynamical and Physical Meteorology," McGraw-Hill Book Company, New York, 1957, p. 55.
31. Harrison, L. P., "The Water Vapor in the Atmosphere over the United States East of the Rocky Mountains," Monthly Weather Review, Vol. 59, 1931, pp. 449-472.
32. \_\_\_\_\_, "Fundamental Concepts and Definitions Relating to Humidity," in "Humidity and Moisture Measurement and Control in Science and Industry," (A. Wexler, Editor-in-Chief), Vol. 3, "Fundamentals and Standards," (A. Wexler and W. A. Wildhack, Editors), Based on papers presented at the 1963 International Symposium on Humidity and Moisture, Washington, D.C., New York: Reinhold Publishing Corporation, 1965, pp. 3-69.
33. Hastenrath, S., "Der Tagesgang des Dampfdrucks in Verschiedenen Höhenlagen der Alpen bei Sommerlichem und Winterlichem Hochdruckwetter," Meteorologische Rundschau, Band 12, Heft 6, 1959, pp. 173-176.
34. Hastenrath, S. L., "The Flux of Atmospheric Water Vapor over the Caribbean Sea and the Gulf of Mexico," Journal of Applied Meteorology, Vol. 5, 1966, pp. 778-788.

35. \_\_\_\_\_, "On General Circulation and Energy Budget in the Area of the Central American Seas," Journal of Atmospheric Sciences, Vol. 23, 1966, pp. 694-711.
36. \_\_\_\_\_, "Diurnal Fluctuation of the Atmospheric Moisture Flux in the Caribbean and Gulf of Mexico Area," Journal of Geophysical Research, Vol. 72, 1967, pp. 4119-4130.
37. Huff, F. A., "Atmospheric Moisture-Precipitation Relations," Journal of the Hydraulics Division, Proceedings of the American Society of Civil Engineers, Vol. 89, No. HY6, Part 1, November 1963, pp. 93-110.
38. Hutcherson, D. C., "Water Vapor in the Atmosphere" in "Humidity and Moisture Measurement and Control in Science and Industry," (Arnold Wexler, Editor-in-Chief), Vol. 2, "Applications," (E. J. Amdur, Editor), Based on papers presented at the 1963 International Symposium on Humidity and Moisture, Washington, D.C., New York: Reinhold Publishing Corporation, 1965, pp. 486-494.
39. Hutchings, J. W., "Water Vapor Flux and Flux Divergence over Southern England: Summer 1954," Quarterly Journal of the Royal Meteorological Society, Vol. 83, 1957, pp. 30-48.
40. \_\_\_\_\_, "Water-vapor Transfer over the Australian Continent," Journal of Meteorology, Vol. 18, 1961, pp. 615-634.
41. Hydrometeorological Section (USWB), "Tables of Precipitable Water and Other Factors for a Saturated Pseudo-Adiabatic Atmosphere," Washington, D.C.: U.S. Government Printing Office, Weather Bureau, Division of Climatological and Hydrologic Services, Hydrometeorological Section, Technical Note No. 14, 1951, 27 pp.
42. Kimball, H. H., "Determination of Atmospheric Turbidity and Water Content," Monthly Weather Review, Vol. 64, 1936, pp. 1-6.
43. \_\_\_\_\_ and I. F. Hand, "The Use of Glass Color Screens in the Study of Atmospheric Depletion of Solar Radiation," Monthly Weather Review, Vol. 61, 1933, pp. 80-83.
44. King, R. L. and H. D. Parry, "Field Tests and Calibration of the Total Atmospheric Water Vapor Hygrometer" in "Humidity and Moisture Measurement and Control in Science and Industry," (A. Wexler, Editor-in-Chief), Vol. 2, "Applications," E. J. Amdur, Editor, Based on papers presented at the 1963 International Symposium on Humidity and Moisture, Washington, D.C., New York: Reinhold Publishing Corporation, 1965, pp. 450-457.
45. Kiss, E. (Compiler), "Annotated Bibliography on Water Vapor Content of the Atmosphere Above 50,000 Ft.," Meteorological and Geostrophysical Abstracts, Vol. 12, August 1961, pp. 1668-1697.

46. Lejay, R. P., "La Mesure à Shanghai de la Quantité Totale de Vapeur d'Eau Contenue dans l'Atmosphère," La Meteorologie, Janvier-Mars 1950, pp. 29-38.
47. List, R. J. (Editor), "Smithsonian Meteorological Tables," Smithsonian Institution, Washington, D.C., Sixth Revised Edition, Seventh Reprint, 1968, pp. 217-223 and 488-492.
48. London, J., "A Study of the Atmospheric Heat Balance," Research Division, College of Engineering, New York University, University Heights, New York, Final Report, Contract No. AF 18(122)-165, 1957, 99 pp.
49. Lorenz, E. N., "The Nature and Theory of the General Circulation of the Atmosphere," World Meteorological Organization, Geneva, WMO-No. 218.TP.115, 1967, 161 pp.
50. Lufkin, D., "Atmospheric Water Vapor Divergence and the Water Balance at the Earth's Surface," Massachusetts Institute of Technology, Cambridge, Mass., Scientific Report No. 4, General Circulation Project, 1959, 44 pp.
51. Manabe, S., "On the Modification of Air-Mass over the Japan Sea when the Outburst of Cold Air Predominates," Journal of the Meteorological Society of Japan, Ser. II, Vol. 35, 1957, pp. 311-326.
52. \_\_\_\_\_, J. Smagorinsky and R. F. Strickler, "Simulated Climatology of a General Circulation Model with a Hydrologic Cycle," Monthly Weather Review, Vol. 93, December 1965, pp. 769-798.
53. Martin, G. E., "Tables of Atmospheric Precipitable Water for Horizontal Paths," U.S. Naval Avionics Facility, Indianapolis 18, Indiana, NAFL-TP-33, BuAer Ted Project No. AVFI-AR-32013, September 1957, 261 pp.
54. Mastenbrook, H. J. and J. E. Dinger, "The Measurement of Water-Vapor Distribution in the Stratosphere," Naval Research Laboratory, Washington, D.C., Report 5551, November 16, 1960, 36 pp.
55. \_\_\_\_\_ and D. R. Purdy, "Vertical Distribution of Water Vapor over Washington, D.C., During 1966 and 1967," Naval Research Laboratory, Washington, D.C., NRL Report 6891, May 27, 1969, 98 pp.
56. McDonald, J. E., "The Evaporation-Precipitation Fallacy," Weather, Vol. 17, 1962, pp. 168-177.
57. McMurry, E. W., "Measurement of Atmospheric Water Vapor by a Spectrophotometric Technique," Institute of Atmospheric Physics, University of Arizona, Tucson, Scientific Report No. 19, ONR Contract No. 2173(02), Sept. 21, 1962, 15 pp. + 15 Figs.

58. Miller, L. E., "The Vertical Distribution of Water Vapor in the Stratosphere and Upper Atmosphere," U.S. Air Force, Air Force Cambridge Research Center, Bedford, Massachusetts, Air Force Surveys in Geophysics No. 45, September 1953, 118 pp.
59. Montefinale, A. C. and H. M. Papée, "Correlation of Precipitable Water Vapour Values over the United States, with Physical and Chemical Parameters. Part I: Seasonal Trends with Precipitation and Its Nitrate Content," Geofisica Pura e Applicata, Vol. 48, 1961-I, p. 203.
60. \_\_\_\_\_, T. W. Zawidzki and H. M. Papée, "Correlation of Precipitable Water Vapour Values over the United States with Physical and Chemical Parameters. Part III: Annual and Seasonal Trends, A Further Approximation," Geofisica Pura e Applicata, Vol. 54, 1963-I, pp. 190-195.
61. Murcray, D. G., F. K. Murcray and W. J. Williams, "Distribution of Water Vapor in the Stratosphere as Determined from Infrared Absorption Measurements," Journal of Geophysical Research, Vol. 67, 1962, pp. 759-766.
62. \_\_\_\_\_, \_\_\_\_\_ and \_\_\_\_\_, "Further Data Concerning the Distribution of Water Vapour in the Stratosphere," Quarterly Journal of the Royal Meteorological Society, Vol. 92, January 1966, pp. 159-161.
63. \_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_ and F. E. Leslie, "Water Vapor Distribution Above 90,000 Feet," Department of Physics, University of Denver, Denver, Scientific Report No. 5 on Contract No. AF 19(604)-2069, AFCRC-TN-60-452, Geophysics Research Directorate, Bedford, Massachusetts, 24 May 1960, 15 pp.
64. Neumann, J., "On the Semi-Diurnal Wave of Atmospheric Vapour Pressure," Archiv für Meteorologie, Geophysik und Bioklimatologie, Ser. B, Vol. 4, 1952, pp. 122-132.
65. Neuwirth, R., "Feuchtegehalt der Atmosphäre und Niederschlagsprognose im Zusammenhang mit der Lage des Beobachtungsortes zum planetarischen Ringstrom (Jet Stream)," Archiv für Meteorologie, Geophysik und Bioklimatologie, Ser. B, Vol. 6, No. 1, 1953, pp. 59-77.
66. Nyberg, A., "A Determination of Some Monthly Values of Evapotranspiration in Finland and in a Region in the Northeastern Atlantic," Geophysica, Vol. 6, No. 3-4, 1958, pp. 377-387.
67. \_\_\_\_\_, "A Computation of the Evaporation in Southern Sweden During 1957," Tellus, Vol. 17, 1965, pp. 473-483.
68. Obasi, G. O. P., "Atmospheric Momentum and Energy Calculations for the Southern Hemisphere During the IGY," Department of Meteorology, Massachusetts Institute of Technology, Cambridge, Scientific Report No. 6, Planetary Circulation Project, Contract No. AF19(604)-6103, AFCRL-62-1128, Air Force Cambridge Research Laboratories, Bedford, Massachusetts, 1 January 1963, 354 pp.

69. Palmén, E., "Evaluation of Atmospheric Moisture Transport for Hydrological Purposes," Secretariat of the World Meteorological Organization, Geneva, Switzerland, International Hydrological Decade (IHD), Report No. 1, 1967, 63 pp.
70. \_\_\_\_\_ and E. Holopainen, "Divergence, Vertical Velocity and Conversion Between Potential and Kinetic Energy in an Extratropical Disturbance," Geophysica, Vol. 8, No. 2, 1962, pp. 89-113.
71. \_\_\_\_\_ and D. Söderman, "Computation of the Evaporation from the Baltic Sea from the Flux of Water Vapour in the Atmosphere," Geophysica, Vol. 8, No. 4, 1966, pp. 261-279.
72. \_\_\_\_\_ and L. A. Vuorela, "On the Mean Circulations in the Northern Hemisphere During the Winter Season," Quarterly Journal of the Royal Meteorological Society, Vol. 89, 1963, pp. 131-138.
73. Papeé, H. M. and A. C. Montefinale, "Correlation of Precipitable Water Values over the United States, with Physical and Chemical Parameters. Part II: Seasonal Trends with Physical Variables," Geofisica Pura e Applicata, Vol. 48, 1961-I, p. 224.
74. Peixoto, J. P., "Hemispheric Humidity Conditions During the Year 1950," Massachusetts Institute of Technology, Cambridge, Scientific Report No. 3, General Circulations Project, 1958, 142 pp.
75. \_\_\_\_\_ and A. R. Crisi, "Hemispheric Humidity Conditions During the IGY," Massachusetts Institute of Technology, Cambridge, Scientific Report No. 6, Planetary Circulations Project, Contract No. AF19(628)-2408, 1965, 166 pp.
76. \_\_\_\_\_ and G. O. P. Obasi, "Humidity Conditions over Africa During the IGY," Massachusetts Institute of Technology, Cambridge, Scientific Report No. 4, Planetary Circulations Project, 1965.
77. \_\_\_\_\_ and B. Saltzman, "Harmonic Analysis of the Mean Northern Hemisphere Water Vapor Distribution for the Year 1950," Geofisica Pura e Applicata, Vol. 41, 1958/III, pp. 183-188.
78. Peterson, K. R., "A Precipitable Water Nomogram," Bulletin of the American Meteorological Society, Vol. 42, February 1961, pp. 119-121.
79. Priestley, C. H. B., "Physical Interactions Between Tropical and Temperate Latitudes," Quarterly Journal of the Royal Meteorological Society, Vol. 77, 1951, pp. 200-214.
80. Rao, K. N., "Regression of Vapour Pressure with Height, Latitude and Longitude," Journal of the Meteorological Society of Japan, 75th Anniversary Volume, November 1957, pp. 342-349.
81. Rasmussen, J. L., "Atmospheric Water Balance of the Upper Colorado River Basin," Department of Atmospheric Science, Colorado State University, Fort Collins, Atmospheric Science Paper No. 121, February 1968, 112 pp.

82. Rasmusson, E. M., "Atmospheric Water Vapor Transport and the Hydrology of North America," Massachusetts Institute of Technology, Cambridge, Report No. A1, Planetary Circulation Project, 1966, 170 pp.
83. \_\_\_\_\_, "Atmospheric Water Vapor Transport and the Water Balance of North America. Part I. Characteristics of the Water Vapor Flux Field," Monthly Weather Review, Vol. 95, 1967, pp. 403-426.
84. \_\_\_\_\_, "Atmospheric Water Vapor Transport and the Water Balance of North America. II. Large-Scale Water Balance Investigations," Monthly Weather Review, Vol. 96, 1968, pp. 720-734.
85. Reitan, C. H., "Surface Dew Point and Water Vapor Aloft," Journal of Applied Meteorology, Vol. 2, 1963, pp. 776-779.
86. \_\_\_\_\_, "Mean Monthly Precipitable Water over the United States, 1946-56," Monthly Weather Review, Vol. 88, No. 1, January 1960, pp. 25-35.
87. \_\_\_\_\_, "Distribution of Precipitable Water Vapor over the Continental United States," Bulletin of the American Meteorological Society, Vol. 41, February 1960, pp. 79-87.
88. Scarborough, J. B., "Numerical Mathematical Analysis," The Johns Hopkins Press, Baltimore, Maryland, 5th Edition, 1962, 594 pp.
89. Schüepp, W., "Die Bestimmung der Komponenten der atmosphärischen Trübungen aus Aktinometermessungen," Archiv für Meteorologie, Geophysik und Bioklimatologie, Ser. B, Vol. 1, 1949, p. 257.
90. Sellers, W. D., "Distribution of Relative Humidity and Dew Point in the Southwestern United States," Institute of Atmospheric Physics, University of Arizona, Tucson, Scientific Report No. 13, February 1, 1960, 49 pp.
91. \_\_\_\_\_, "Physical Climatology," The University of Chicago Press, Chicago, 1965, 272 pp.
92. Shands, A. L., "Mean Precipitable Water in the United States," Hydro-meteorological Section, Division of Climatological and Hydrologic Services, Weather Bureau, Washington, D.C., Technical Paper No. 10, 1949, 48 pp.
93. Showalter, A. K., "Precipitable Water Template," Bulletin of the American Meteorological Society, Vol. 35, 1954, pp. 129-131.
94. Shuman, F. G. and J. B. Hovermale, "An Operational Six-Layer Primitive Equation Model," Journal of Applied Meteorology, Vol. 7, 1968, pp. 525-547.
95. Sissenwine, N. and M. Gutnick, "Precipitable Water Along High Altitude Ray Paths," Proceedings of Infrared Information Symposium, Vol. 5, No. 3, July 1960, pp. 5-12.
96. Smagorinsky, J. and G. D. Collins, "On Numerical Prediction of Precipitation," Monthly Weather Review, Vol. 83, 1955, pp. 53-67.

97. Smith, W. L., "Note on the Relationship Between Total Precipitable Water and Surface Dew Point," Journal of Applied Meteorology, Vol. 5, 1966, pp. 726-727.
98. Solomatina, I. I., "The Diurnal Variation of the Absolute Humidity in the Atmosphere Boundary Layer," Air Force Cambridge Research Laboratories, Bedford, Massachusetts, Translation from the Russian, T-R-384, September 1965, 11 pp.
99. Solot, S. B., "Computation of Depth of Precipitable Water in a Column of Air," Monthly Weather Review, Vol. 67, 1939, pp. 100-103.
100. Spar, J., "A Suggested Technique for Quantitative Precipitation Forecasting," Monthly Weather Review, Vol. 81, 1953, pp. 217-221.
101. Starr, V. P. and J. P. Peixoto, "On the Global Balance of Water Vapor and the Hydrology of Deserts," Tellus, Vol. 10, 1958, pp. 188-194.
102. \_\_\_\_\_ and \_\_\_\_\_, "On the Zonal Flux of Water Vapor in the Northern Hemisphere," Geofisica Pura e Applicata, Vol. 47, 1960/III, pp. 199-203.
103. \_\_\_\_\_ and \_\_\_\_\_, "The Hemispheric Eddy Flux of Water Vapor and Its Implications for the Mechanics of the General Circulation," Archiv für Meteorologie, Geophysik und Bioklimatologie, Vol. 14, 1964, pp. 111-130.
104. \_\_\_\_\_, \_\_\_\_\_ and A. R. Crisi, "Hemispheric Water Balance for the IGY," Tellus, Vol. 17(4), 1965, pp. 463-472.
105. \_\_\_\_\_, \_\_\_\_\_ and G. C. Livadas, "On the Meridional Flux of Water Vapor in the Northern Hemisphere," Geofisica Pura e Applicata, Vol. 39, 1958, pp. 174-185.
106. \_\_\_\_\_ and R. M. White, "Balance Requirements of the General Circulation of the Atmosphere," Geophysics Research Directorate, Cambridge, Massachusetts, Geophysical Research Papers, No. 35, 1954.
107. \_\_\_\_\_ and \_\_\_\_\_, "Direct Measurements of the Hemispheric Poleward Flux of Water Vapor," Journal of Marine Research, Vol. 14, 1955, pp. 217-225.
108. Sutcliffe, R. C., "Water Balance and the General Circulation of the Atmosphere," Quarterly Journal of the Royal Meteorological Society, Vol. 82, October 1956, pp. 385-395.
109. The Travelers Research Center, Inc., "Moisture Flux over North America. Part I: Tabulation of Monthly Values of Vertically Integrated Moisture and Moisture Transport for North America, May 1958 - April 1963. Volume A: 0000 GMT; Volume B: 1200 GMT; Volume C: Unweighted Mean of 0000 + 1200 GMT," The Travelers Research Center, Inc., Hartford, Connecticut, Final Report 7477-244, Contract Cwb-11313, for ESSA, Weather Bureau, December 1966, pp. A-240 + B-240 + C-240.

110. Tuller, S. E., "World Distribution of Mean Monthly and Annual Precipitable Water," Monthly Weather Review, Vol. 96, November 1968, pp. 785-796.
111. Tunnell, G. A., "World Distribution of Atmospheric Water Vapour Pressure," Great Britain Meteorological Office, Air Ministry, London, Geophysical Memoirs No. 100, Vol. XII, 1958, 61 pp.
112. Väisänen, A., "A Computation of the Evaporation over Finland During a Rainless Period Based on the Divergence of the Water-Vapor Flux," Geophysica, Vol. 8, 1962, pp. 159-165.
113. Van de Boogaard, H. M. E., "A Preliminary Investigation of the Daily Meridional Transfer of Atmospheric Water Vapour Between the Equator and 40°N.," Tellus, Vol. 16, 1964, pp. 43-54.
114. Vassy, A. and E., "Recherches sur l'épaisseur d'eau condensable de l'atmosphère," Journal Scientifique de la Meteorologie, 2<sup>e</sup> Année, No. 7, Juli-Septembre 1950, pp. 65-74.
115. Vuorela, L. A. and I. Tuominen, "On the Mean Zonal and Meridional Circulations and the Flux of Moisture in the Northern Hemisphere During the Summer Season," Pure and Applied Geophysics, Basel, Vol. 57, 1964, pp. 167-180.
116. White, R. M., "The Meridional Eddy Flux of Energy," Quarterly Journal of the Royal Meteorological Society, Vol. 77, 1951, pp. 188-199.
117. Wilson, R. K., "An Attempt to Measure the Water Vapor Divergence over Greenland by Direct Application of Radiosonde Data," Department of Meteorology and Climatology, University of Washington, Seattle, Scientific Report No. 3, May 1959, 38 pp.
118. Younkin, R. J., J. A. LaRue and F. Sanders, "The Objective Prediction of Clouds and Precipitation Using Vertically Integrated Moisture and Adiabatic Vertical Motions," Journal of Applied Meteorology, Vol. 4, 1965, pp. 3-17.

## ANNEX 1

THEORY UNDERLYING FORMULAS FOR COMPUTING  
PRECIPITABLE WATER

Consider a sample of atmospheric air in the form of a cylindrical vertical column whose horizontal base rests on the earth's surface and whose top occurs at some prescribed level in the free air, such as the 500-mb level. For the sake of simplicity we shall assume that the atmospheric air contains water vapor but no condensed phases of water, either liquid or solid. We also disregard the presence of aerosols. To facilitate visualization of the problem, one may think that the horizontal cross-section area  $A_c$  of the vertical column has a convenient, specific value; for example,  $A_c = 1 \text{ m}^2$ .

Let  $\rho_v$  = the absolute humidity - that is, the mass of water vapor per unit volume of moist air at geometric height  $Z$  above sea level within the air column. We assume that the concentration of vapor content within the air column is horizontally homogeneous; so that its vertical distribution may be considered to be represented by a single-valued function of geopotential  $H$  or pressure  $P$ . In this approach time is taken constant.

Now, let us take a horizontal slice of infinitesimal thickness  $dZ$  through the vertical column of air such that the lower boundary of the slice is at geometric altitude  $(Z - dZ/2)$  and its upper boundary at geometric altitude  $(Z + dZ/2)$ . Let  $dm$  = the mass of water vapor contained within this slice inside the cylindrical wall of the vertical column; then

$$dm = \rho_v \cdot A_c \cdot dZ \quad (A1)$$

Now, suppose that all of the water vapor of mass  $dm$  within the aforementioned slice of the air column was condensed into liquid water and that the entire mass  $dm$  was precipitated on the horizontal base of the air column where it was retained by vertical walls.

Let  $\rho_w$  = the density of the liquid water precipitated on the base of the vertical column and let  $dW$  = the thickness of the layer of water of mass  $dm$  and density  $\rho_w$  thus deposited. Then  $dW$  will be given by the relation

$$dm = \rho_w \cdot A_c \cdot dW \quad (A2)$$

On equating the last two expressions and solving for  $dW$ , one obtains

$$dW = \left( \frac{\rho_v}{\rho_w} \right) dZ \quad (A3)$$

provided that consistent units are employed; e.g.,  $\rho_v$  and  $\rho_w$  in grams/m<sup>3</sup>, and  $Z$  and  $W$  in meters.

Now let  $H$  = geopotential, in geopotential meters (gpm), corresponding to the geometric height  $Z$  and latitude  $\phi$  at the point in the free air at which the mass  $dm$  of water vapor exists within the slice. The relationship between differentials of  $Z$  and  $H$  is

$$dZ = \left( \frac{g_0}{g} \right) dH \quad (A4)$$

considering that, by definition,  $1 \text{ gpm} = 9.8 \text{ m}^2/\text{sec}^2$

where

$g$  = acceleration of gravity (in m/sec<sup>2</sup>);

$g_0$  = dimensional constant =  $9.8 \text{ m}^2/\text{sec}^2 \cdot \text{gpm}$ ;

$Z$  = geometric height with reference to sea level (in meters);

$H$  = geopotential (in gpm).

Let  $H_s$  = the geopotential at the earth's surface where the base of the vertical air column rests, and let  $H_t$  = the geopotential at the top of the vertical air column whose precipitable water is being computed. Then, on substituting Eq. (A4) in Eq. (A3) and integrating the resulting expression between the limits of  $H_s$  and  $H_t$ , we obtain the precipitable water (in meters depth)

$$W = \left( \frac{g_0}{\rho_w} \right) \int_{H_s}^{H_t} \left( \frac{\rho_v}{g} \right) dH \quad (A5)$$

In deriving this equation we assume that the density of the liquid water precipitated is a constant.

We may make the assumption that at ordinary rainwater temperatures the density of the water will be represented by

$$\rho_w = 10^6 \text{ grams/m}^3 . \quad (\text{A6})$$

Now, if we further assume that the water vapor in the moist air mixture obeys Dalton's law of partial pressures and the ideal-gas law, we may write [32]

$$\rho_v = \left( \frac{M_v}{R^*} \right) \left( \frac{e}{T} \right) \quad (\text{A7})$$

where

$M_v$  = the gram molecular weight of water vapor = 18.01534 grams/mole;

$R^*$  = universal gas constant =  $8.31432 \times 10^{-2}$  mb·m<sup>3</sup>/mole·°K;

$e$  = partial pressure of water vapor in the moist air mixture (millibars);

$T$  = absolute thermodynamic temperature (°K).

The substitution of Eq. (A7) in Eq. (A5) yields

$$W = \left( \frac{g_0 M_v}{\rho_w R^*} \right) \int_{H_s}^{H_t} \left( \frac{e}{gT} \right) dH \quad \text{in meters} \quad (\text{A8})$$

Since we wish the precipitable water to be expressed in units of hundredths of an inch (0.01 in.), we must use the conversion factor

$$\frac{10^4 (0.01 \text{ in.})}{2.54 \text{ m.}} = 1 . \quad (\text{A9})$$

On introducing Eqs. (A6) and (A9) into the right-hand side of Eq. (A8) and substituting the known values of the constants  $g_0$ ,  $M_v$  and  $R^*$ , Eq. (A8) may be written in the form

$$W = 8.36 \left( \frac{\text{m} \cdot ^\circ\text{K}}{\text{sec}^2 \cdot \text{mb} \cdot \text{gpm}} \right) (0.01 \text{ inch}) \int_{H_s}^{H_t} \left( \frac{e}{gT} \right) dH \quad (\text{A10})$$

where the vapor pressure  $e$  must be expressed in millibars (mb), the absolute temperature  $T$  in degrees Kelvin ( $^\circ\text{K}$ ), the acceleration of gravity  $g$  in m/sec<sup>2</sup>, and the geopotentials  $H$ ,  $H_s$  and  $H_t$  in gpm. Under these conditions the coefficient of Eq. (A10) and the identical Eq. (3.1) has dimensions such that the value of the precipitable water  $W$  will come out in units of hundredths of an inch (0.01 in.).

Now, suppose that with reference to Eqs. (A8) and (A10), we make the assumption that the acceleration of gravity  $g$  is a specific constant, namely,

$$g = 9.8 \text{ m/sec}^2 . \quad (\text{A11})$$

When Eq. (A11) is substituted in Eq. (A10), it becomes

$$W = 0.853 \left( \frac{^\circ\text{K}}{\text{mb} \cdot \text{gpm}} \right) (0.01 \text{ inch}) \int_{H_s}^{H_t} \left( \frac{e}{T} \right) dH \quad (\text{A12})$$

where, as in the case of Eq. (A10), the vapor pressure  $e$  must be expressed in millibars (mb), the absolute temperature  $T$  in degrees Kelvin ( $^\circ\text{K}$ ), and the geopotentials  $H$ ,  $H_s$  and  $H_t$  in gpm. Like the parallel case treated previously, that is Eqs. (A10) and (3.1), the coefficient of Eq. (A12) and the identical Eq. (3.2) has dimensions such that the value of the precipitable water  $W$  will be expressed in units of hundredths of an inch (0.01 in.).

Now we wish to consider the problem of computing  $W$  when the pressure  $P$  is employed as the independent variable instead of the geopotential  $H$ . To this end we first write the hydrostatic equation in differential form pertinent to the horizontal slice of thickness  $dZ$ :

$$dP = -\rho g dZ \quad (A13)$$

where

$dP$  = difference of pressure between bottom and top of slice;

$\rho$  = the density of the moist air mixture.

When Eq. (A4) is substituted in Eq. (A13), it yields

$$dH = -\frac{dP}{\rho g_0} \quad (A14)$$

By substituting Eq. (A14) in Eq. (A5) and transforming the lower and upper limits of the integral ( $H_s$  and  $H_t$ ) to the corresponding pressure parameters ( $P_s$  and  $P_t$ ), we obtain

$$W = -\left(\frac{1}{\rho_w}\right) \int_{P_s}^{P_t} \left(\frac{\rho_v}{g\rho}\right) dP \quad (\text{in units of } Z) \quad (A15)$$

where

$P_s$  = the barometric pressure at the earth's surface at the base of the vertical column of atmospheric air whose precipitable water  $W$  is being computed;

$P_t$  = the barometric pressure at the top of the above-mentioned vertical column.

Let  $q$  = specific humidity (in grams of water vapor per gram of moist air). On the basis of the definition of specific humidity we may write

$$q = \left(\frac{\rho_v}{\rho}\right) = \frac{(M_v/M_a)e}{P - [1 - (M_v/M_a)]e} \quad (A16)$$

where

$M_v$  = gram-molecular weight of water vapor;

$M_a$  = apparent gram-molecular weight of clean, dry, atmospheric air.

Since  $M_v = 18.01534$  grams/mole, and  $M_a = 28.9645$  grams/mole [32], we find

$(M_v/M_a) = 0.62198$ , and  $[1 - (M_v/M_a)] = 0.37802$ , the values of which may now be introduced in Eq. (A16).

On this basis the substitution of Eq. (A16) in Eq. (A15) gives

$$W = - \left( \frac{0.62198}{\rho_w} \right) \int_{P_s}^{P_t} \frac{e}{g(P - 0.37802e)} dP \quad (A17)$$

where, again, W is expressed in the same unit as Z, namely, the meter.

Since the ratio  $(e/P)$  is dimensionless, one may deduce from Eq. (A17) that the dimensions of P must be determined by the following identity, in order to maintain dimensional homogeneity:

$$(\text{Dimension of } W) = \frac{(\text{Dimensions of } P)}{(\text{Dimensions of } \rho_w) \times (\text{Dimensions of } g)}$$

Thus, when W is in meters,  $\rho_w$  in  $\text{grams}/\text{m}^3$ , and g in  $\text{m}/\text{sec}^2$ , one finds that the corresponding dimensions of P in Eq. (A17) would be obtainable by means of the relationship

$$(\text{Dimensions of } P) = \left( \frac{\text{gram} \cdot \text{m}}{\text{m}^2 \cdot \text{sec}^2} \right) = \text{unit of } P \text{ involved in Eq. (A17)} \quad (A18)$$

The foregoing relationship indicates that P in Eq. (A17) is expressed in terms of an unusual unit. Instead, we should like it to be expressed in terms of the millibar because that is the conventional pressure unit used in radio-sonde work.

To do this we note that, by definition,

$$1 \text{ mb} = 1,000 \text{ dynes}/\text{cm}^2$$

where

$$1 \text{ dyne} = 1 \text{ gram} \cdot \text{cm}/\text{sec}^2$$

Thus, on combining these relationships, we find

$$1 \text{ mb} = 10^3 \frac{\text{dynes}}{\text{cm}^2} = 10^3 \left( \frac{\text{gram} \cdot \text{cm}}{\text{sec}^2 \cdot \text{cm}^2} \right) = 10^5 \left( \frac{\text{gram} \cdot \text{m}}{\text{m}^2 \cdot \text{sec}^2} \right) \quad (\text{A19})$$

Therefore, we deduce from the last two equations that

$$(\text{dP}) \text{ in units of } \left( \frac{\text{gram} \cdot \text{m}}{\text{m}^2 \cdot \text{sec}^2} \right) = 10^5 \times (\text{dP}) \text{ in mb units} \quad (\text{A20})$$

or we can express the foregoing relationships in the form of a conversion factor:

$$\frac{10^5 \left( \frac{\text{gram} \cdot \text{m}}{\text{m}^2 \cdot \text{sec}^2} \right)}{\text{mb}} = 1. \quad (\text{A21})$$

Now we wish to transform Eq. (A17) so that it will have a more directly usable character. To do this one may take three appropriate steps: (a) substitute in the equation the value of the liquid water density  $\rho_w$  indicated by Eq. (A6); (b) introduce in the right-hand side of the equation the conversion factor shown in Eq. (A9) for the purpose of converting  $W$  from units of  $Z$  (meters) to units of hundredths of an inch; and (c) also introduce the conversion factor contained in Eq. (A21) with a view to converting pressures to millibars from the awkward unit indicated on the left-hand side of Eq. (A20). On this basis we may rewrite Eq. (A17) in the form

$$W = -244.9 \left( \frac{\text{m}}{\text{sec}^2 \cdot \text{mb}} \right) (0.01 \text{ inch}) \int_{P_s}^{P_t} \frac{e}{g(P - 0.37802e)} dP \quad (\text{A22})$$

where the precipitable water  $W$  is now expressed in hundredths of an inch (0.01 in.),  $g$  is in  $\text{m/sec}^2$ , and all pressure parameters ( $e$ ,  $P$ ,  $P_s$  and  $P_t$ ) are in millibars (mb). This is the basis of Eq. (4.1).

Finally, in case we wish to make the assumption that the acceleration of gravity  $g$  is a specific constant having the value stipulated by Eq. (A11), we may substitute that expression in Eq. (A22) which then becomes

$$W = -24.99 \left( \frac{1}{\text{mb}} \right) (0.01 \text{ inch}) \int_{P_s}^{P_t} \frac{e}{(P - 0.37802e)} dP \quad (\text{A23})$$

where the value of  $W$  is given in units of 0.01 inch, while the values of the pressure parameters ( $e$ ,  $P$ ,  $P_s$ ,  $P_t$ ) are still in millibars. This is the origin of Eq. (4.2).

The following points should be noted: (A) both Eqs. (A12) and (A23) are approximations owing to the assumption underlying Eq. (A11) which was used in deriving those relationships; (B) all of the final expressions for  $W$  require the employment of variables and parameters in terms of certain specified units in order to satisfy the demands of dimensional homogeneity. The latter, (B), implies that when the value of any parameter or variable is substituted in Eqs. (3.1), (3.2), (4.1), (4.2) or the equivalent Eqs. (A10), (A12), (A22), (A23), the precise units of the parameter or variable should be included.

Two further remarks in the nature of reminders will be germane: None of the specified equations include any effects due to the possible presence of aerosols, and liquid water or solid water in suspension; and the extremely slight effect of the divergence of the vertical lines above the earth's surface has been neglected.

(ESA)  
 OUTLINE OF METHOD OF DETERMINING THE ACCELERATION OF  
 GRAVITY  $g$  AS A FUNCTION OF LATITUDE AND GEOPOTENTIAL

It will be noted that the acceleration of gravity  $g$  is treated as a variable of the problem in Eqs. (3.1) and (4.1), or the equivalent Eqs. (A10) and (A22).

In connection with Eqs. (3.1) and (A10) the latitude  $\phi$  is a constant parameter of the problem while  $H$ , the geopotential, is taken as the independent variable. Therefore, in evaluating those equations numerically one needs relationships which would permit the value of  $g$  to be expressed as a function of  $\phi$  and  $H$ . To this end we will indicate the pertinent relationships in accordance with the specifications given in the Smithsonian Meteorological Tables [47].

Let

- $g$  = acceleration of gravity, in  $\text{m/sec}^2$ ;
- $Z$  = geometric altitude (height above sea level), in meters;
- $H$  = geopotential, in  $\text{gpm}$  (see Ref. 47, page 217);
- $\phi$  = latitude of the station;
- $g_\phi$  = the acceleration of gravity at sea level at the latitude of the station, in  $\text{m/sec}^2$ .

According to the information given on pages 490-491 of Ref. 47, one may write for the theoretical value of the acceleration of gravity at sea level at the latitude of the station

$$g_\phi = 9.80616(1 - 0.0026373 \cos 2\phi + 0.0000059 \cos^2 2\phi) \text{ in } \text{m/sec}^2 \quad (\text{B1})$$

The acceleration of gravity  $g$  at a point in the free air characterized by geometric altitude  $Z$  and latitude  $\phi$  differs from  $g_\phi$ , the theoretical value of the acceleration of gravity at mean sea level at the latitude of the station, by certain correction terms which vary with both  $Z$  and  $\phi$ . Therefore, we may represent  $g$  by combining  $g_\phi$  with the correction terms as indicated below (see Ref. 47, page 490):

$$\begin{aligned}
 g = & 9.80616(1 - 0.0026373 \cos 2\phi + 0.0000059 \cos^2 2\phi) \\
 & - (3.085462 \times 10^{-6} + 2.27 \times 10^{-9} \cos 2\phi)Z \\
 & + (7.254 \times 10^{-13} + 1.0 \times 10^{-15} \cos 2\phi)Z^2 \\
 & \text{in m/sec}^2, \text{ when } Z \text{ is in meters.}
 \end{aligned} \tag{B2}$$

Combination of the last two terms in this expression represents the correction mentioned in the previous paragraph. An additional term is given in Ref. 47, page 490, but it is considered likely that two terms will suffice for present purposes.

The geometric altitude  $Z$  is itself a function of the latitude  $\phi$  and the geopotential  $H$ , as indicated by the following expression:

$$Z = \frac{L_1 H}{(K_1 - H)} \tag{B3}$$

where  $L_1$  and  $K_1$  are functions of the latitude  $\phi$  tabulated in table 4. The same functions are also tabulated on page 219 of Ref. 47, in terms of different symbols. An outline of a proof of Eq. (B3) will be found on pages 217-218 of Ref. 47.

It follows from the information given in the foregoing that one may substitute Eq. (B3) in Eq. (B2) and thus obtain an expression which represents  $g$  as a function of  $\phi$  and  $H$ .

Analytical expressions for  $L_1$  and  $K_1$  are presented below. According to Eq. (6) on page 218 of Ref. 47, the function  $L_1$  may be written

$$L_1 = \frac{2g_\phi}{-\left(\frac{\partial g}{\partial Z}\right)_{Z=0}} \tag{B4}$$

where  $g_\phi$  is represented analytically by Eq. (B1) and the term in the denominator of Eq. (B4) is determined by the relationship

$$\begin{aligned}
 -\left(\frac{\partial g}{\partial Z}\right)_{Z=0} = & (3.085462 \times 10^{-6} + 2.27 \times 10^{-9} \cos 2\phi \\
 & - 2 \times 10^{-12} \cos 4\phi) \text{ in sec}^{-2}
 \end{aligned} \tag{B5}$$

The function  $K_1$  is defined by

$$K_1 = \left( \frac{g_\phi \cdot L_1}{9.8} \right) \quad \text{in gpm} \quad (\text{B6})$$

where  $g_\phi$  is given by Eq. (B1) and  $L_1$  is represented by Eq. (B4).

Examination of the foregoing reveals that the system of Eqs. (B1) - (B6) permits one to express  $g$  as a function of  $\phi$  and  $H$ ; that is, in symbolic form

$$g = F_1[H, \phi]. \quad (\text{B7})$$

Such a function may then be substituted in Eq. (3.1) or its counterpart Eq. (A10) as an expression for  $g$ , when it is treated as a variable of the problem.

However, in the case relating to Eqs. (4.1) and (A22), one must deal with pressure  $P$  as the independent variable. For this case the geopotential  $H$  is considered as a function of  $P$ :

$$H = f(P) \quad (\text{B8})$$

where the dependence of  $H$  on  $P$  is determined from the tabulated data relating to the significant levels and the standard isobaric surfaces. That is, the geopotentials for these various levels are computed on the basis of the known pressures pertaining to the levels together with other data such as temperatures and dew-point values.

Substitution of Eq. (B8) in Eq. (B7) yields

$$g = F_1[f(P), \phi] \quad (\text{B9})$$

so that  $g$  could be expressed as a function of  $P$  and  $\phi$ .

Table 4

Table Giving the Parameters  $L_1$  and  $K_1$   
as Functions of the Latitude

Latitude	$L_1$ m	$K_1$ gpm	Latitude	$L_1$ m	$K_1$ gpm	Latitude	$L_1$ m	$K_1$ gpm
0°	6334984	6322289	30°	6345653	6341274	60°	6367103	6379519
1	6334995	6322306	31	6346305	6342440	61	6367738	6380658
2	6335035	6322378	32	6346967	6343616	62	6368371	6381783
3	6335099	6322492	33	6347647	6344829	63	6368983	6382880
4	6335191	6322654	34	6348337	6346058	64	6369582	6383945
5	6335306	6322863	35	6349033	6347300	65	6370171	6385001
6	6335449	6323114	36	6349736	6348549	66	6370732	6386003
7	6335616	6323407	37	6350456	6349834	67	6371280	6386979
8	6335806	6323745	38	6351177	6351120	68	6371810	6387931
9	6336030	6324144	39	6351907	6352422	69	6372324	6388847
10	6336267	6324570	40	6352638	6353718	70	6372821	6389734
11	6336536	6325048	41	6353376	6355034	71	6373294	6390578
12	6336824	6325557	42	6354120	6356363	72	6373743	6391379
13	6337140	6326120	43	6354868	6357696	73	6374175	6392150
14	6337480	6326725	44	6355612	6359025	74	6374584	6392879
15	6337838	6327368	45	6356360	6360358	75	6374972	6393574
16	6338219	6328046	46	6357108	6361685	76	6375340	6394230
17	6338626	6328769	47	6357852	6363015	77	6375680	6394839
18	6339058	6329537	48	6358601	6364349	78	6375997	6395406
19	6339506	6330333	49	6359345	6365679	79	6376293	6395938
20	6339971	6331158	50	6360083	6366996	80	6376562	6396412
21	6340461	6332028	51	6360820	6368313	81	6376806	6396855
22	6340973	6332945	52	6361552	6369612	82	6377025	6397240
23	6341496	6333874	53	6362273	6370900	83	6377222	6397597
24	6342043	6334845	54	6362993	6372188	84	6377389	6397892
25	6342603	6335848	55	6363697	6373446	85	6377532	6398151
26	6343184	6336879	56	6364401	6374699	86	6377654	6398369
27	6343782	6337946	57	6365092	6375932	87	6377746	6398531
28	6344393	6339032	58	6365772	6377148	88	6377811	6398647
29	6345018	6340145	59	6366435	6378328	89	6377845	6398707
						90	6377862	6398737



(Continued from inside front cover)

- WBTM TDL 16 Objective Visibility Forecasting Techniques Based on Surface and Tower Observations. Donald M. Gales, October 1968. (PB-180 479)
- WBTM TDL 17 Second Interim Report on Sea and Swell Forecasting. N. A. Pore and Lt. W. S. Richardson, USESSA, January 1969. (PB-182 273)
- WBTM TDL 18 Conditional Probabilities of Precipitation Amounts in the Conterminous United States. Donald L. Jorgensen, William H. Klein, and Charles F. Roberts, March 1969. (PB-183 144)
- WBTM TDL 19 An Operationally Oriented Small-Scale 500-Millibar Height Analysis. Harry R. Glahn and George W. Hollenbaugh, March 1969. (PB-184 111)
- WBTM TDL 20 A Comparison of Two Methods of Reducing Truncation Error. Robert J. Bermowitz, May 1969. (PB-184 741)
- WBTM TDL 21 Automatic Decoding of Hourly Weather Reports. George W. Hollenbaugh, Harry R. Glahn, and Dale A. Lowry, July 1969. (PB-185 806)
- WBTM TDL 22 An Operationally Oriented Objective Analysis Program. Harry R. Glahn, George W. Hollenbaugh, and Dale A. Lowry, July 1969. (PB-186 129)
- WBTM TDL 23 An Operational Subsynoptic Advection Model. Harry R. Glahn, Dale A. Lowry, and George W. Hollenbaugh, July 1969. (PB-186 389)
- WBTM TDL 24 A Lake Erie Storm Surge Forecasting Technique. William S. Richardson and N. Arthur Pore, August 1969. (PB-185 778)
- WBTM TDL 25 Charts Giving Station Precipitation in the Plateau States From 850- and 500-Millibar Lows During Winter. August F. Korte, Donald L. Jorgensen, and William H. Klein, September 1969. (PB-187 476)
- WBTM TDL 26 Computer Forecasts of Maximum and Minimum Surface Temperatures. William H. Klein, Frank Lewis, and George P. Casely, October 1969. (PB-189 105)
- WBTM TDL 27 An Operational Method for Objectively Forecasting Probability of Precipitation. Harry R. Glahn and Dale A. Lowry, October 1969. (PB-188 660)
- WBTM TDL 28 Techniques for Forecasting Low Water Occurrences at Baltimore and Norfolk. Lt. (jg) James M. McClelland, USESSA, March 1970. (PB-191 744)
- WBTM TDL 29 A Method for Predicting Surface Winds. Harry R. Glahn, March 1970. (PB-191 745)
- WBTM TDL 30 Reserved.
- WBTM TDL 31 Reserved.
- WBTM TDL 32 Reserved.



1  
1



1  
1

